Single trace integral formulations of second kind for acoustic scattering with metallic domain

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Abstract

We study the scattering of acoustic waves by an object composed of several adjacent parts with different material properties. One of the part is an impenetrable metallic domain. For this problem we derive an integral formulation of the second kind. This formulation only involves one Dirichlet datum and one Neumann datum at each point of each interface of the object.

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The simulation of wave propagation in a medium with piecewise constant wave number has several apllications related to acoustics and electromagnetics. To study this type of problems, one possible approach is to transform partial derivate equations (PDEs) into boundary integral equations (BIEs). Most of the literature about BIEs deals with objects composed at most of two parts with different material properties. However, in practice, the geometrical configuration implies to study the case with three or more different parts adjacent to each other : the multiple subdomains scattering.

About multiple subdomains scattering, von Petersdorff proposed a formumlation of first kind for scalars problems in [7], extended by Buffa to Maxwell's equations in [2]. In these formulations, transmission conditions are taken in account via the choice of variational spaces and it only involves one Dirichlet datum and one Neumann datum at each point of each interface of the object. These formulations are called single trace formulations of first kind. However they aren't well conditionned and no efficient conditionner has been proposed for them.

More recently, in [5], Hiptmair and Jerez-Hanckes developped another integral formulation of first kind for multiple subdomains scattering, with good properties in terms of preconditionning but the preconditionning requires the solution to integral equations local to each subdomain. They called this formulation multitrace formulation of first kind, as all unknows of the problem are doubled on each interface.

Claeys proposed first a single trace formulation of second kind in [3] for multiple subdomains scattering. Because it is of second kind, his formulation is intrisically well conditionned.

In this report, we study the scattering of acoustic waves by an object composed of several adjacent parts with different material properties and one of the part is an impenetrable metallic domain : we call it multiple subdomains scattering with metallic domain. We propose a single trace formulation of second kind. This work is the continuity of the work done by Claeys in [3].

We describe first the problem we consider in Section 1. Then we introduce the adapted functionnal spaces to study our problem in Section 2. We propose in Section 3 a single trace formulation of second kind for scattering by one isolated metallic domain and we test it numerically in 2-D. We delevoppe finally in Section 4 a single trace formulation of second kind for multiple subdomains scattering with one metallic domain and we test it numerically in 2-D.

1 Setting of the problem

We consider a partition $\mathbb{R}^d = \cup_{i=0}^{n+1}\overline{\Omega}_i$ where $\cup_{i=1}^{n+1}\overline{\Omega}_i$ is bounded : Ω_0 is the exterior domain and Ω_{n+1} is the metallic domain. Each Ω_i is a connnected open Lipschitz subset.
We also set $\Omega = \mathbb{R}^d \setminus \overline{\Omega_{n+1}}$ ($\partial\Omega = \partial\Omega_{n+1}$) and $\Gamma = \cup_{i=0}^{n+1}\partial\Omega_i$. Note that there may exist points where three or more sub-domains would be contiguous, which is precisely the situation that we wish to tackle. For each i the vector n_i refers to the normal vector on $\partial\Omega_i$ directed toward the exterior of Ω_i.

The problem that we study Let $u_{\mathrm{inc}} \in \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d)$ satisfy $\Delta u_{\mathrm{inc}} + \kappa_0^2 u_{\mathrm{inc}} = 0$ in $\mathbb{R}^d$ for some $\kappa_0 \in \mathbb{R}$. This function plays the role of incident field. In the present report we study the following problem:

$$\text{Find } \; u \in \mathrm{H}^1_{0,\mathrm{loc}}(\Delta, \Omega) \; \text{ such that} \tag{1}$$

$$\begin{cases} \Delta u + \kappa_i^2 u = 0 \quad \text{in } \; \Omega_i \;, \quad i = 0 \dots n \\ u - u_{\mathrm{inc}} \; \text{ outgoing radiating in } \Omega_0 \end{cases} \tag{2}$$

where each $\kappa_i \in \mathbb{R}_+$ refers to the wave number inside Ω_i. In Equation (2), the outgoing radiation condition refers to the standard Sommerfeld radiation condition, see [4, 6]. It can be reformulated as

$$\lim_{r\to\infty} \int_{\partial \mathrm{B}_r} |\partial_r u - i\kappa_0 u|^2 d\partial\Omega_r = 0 \quad \text{with} \quad r = |\mathbf{x}| \; . \tag{3}$$

where $\mathrm{B}_r = \{\mathbf{x} \in \mathbb{R}^d |\; |\mathbf{x}| < r\}$. Transmission conditions are imposed through Equation (1). Problem (1)-(2) is a mix of transmission problems for $\cup_{i=0}^{n}\Omega_i$ and impenetrable problem for Ω_{n+1} (modelised by an homogeneous Dirichlet condition on $\partial\Omega = \partial\Omega_{n+1}$).

2 Adapted functionnal spaces

We present here the right spaces for Problem (1)-(2) : the Multi and Single trace spaces.

Multi trace space In order to reformulate Equation (2) as an integral equation posed over Γ, a natural functional setting consists in taking cartesian products of trace spaces, namely

$$\mathbb{H}(\Gamma) = \prod_{j=0}^{n} \Big[\mathrm{H}^{\frac{1}{2}}(\partial\Omega_j) \times \mathrm{H}^{-\frac{1}{2}}(\partial\Omega_j)\Big] \quad \text{equipped with the norm}$$

$$\|U\| = \Big(\sum_{j=0}^{n} \|u_j\|^2_{\mathrm{H}^{\frac{1}{2}}(\partial\Omega_j)} + \|p_j\|^2_{\mathrm{H}^{-\frac{1}{2}}(\partial\Omega_j)}\Big)^{\frac{1}{2}} \quad \text{when } \; U = (u_j, p_j)_{0\le j\le n}$$

It is a Hilbert space. Note that the maximum index is n and not $n+1$ ans observe that this space can be identified to its own dual by means of the following duality pairing

$$\mathrm{B}(U, V) = \sum_{i=0}^{n} \int_{\partial\Omega_i} u_i\, \overline{q_i}\, d\sigma - \int_{\partial\Omega_i} p_i\, \overline{v_i}\, d\sigma$$
$$\text{where} \quad U = (u_j, p_j)_{0\le j\le n} \in \mathbb{H}(\Gamma) \quad \text{and} \quad V = (v_j, q_j)_{0\le j\le n} \in \mathbb{H}(\Gamma)$$

This bilinear form is non-degenerate: if $\mathrm{B}(U, V) = 0 \;\forall V \in \mathbb{H}(\Gamma)$ then $U = 0$.

Single trace spaces Now we introduce spaces that seem more adapted to the treatment of transmission conditions. This setting is inspired by [1]. We set

$$
\begin{aligned}
\mathbb{X}^{+\frac{1}{2}}(\Gamma) &= \Big\{ (v_i) \in \Pi_{i=0}^{n} \mathrm{H}^{\frac{1}{2}}(\partial\Omega_i) \;\Big|\; \exists v \in \mathrm{H}^1_{\mathrm{loc}}(\Omega) \text{ with } v|_{\partial\Omega_j} = v_j,\ \forall j = 0\dots n \Big\}\\
\mathbb{X}^{-\frac{1}{2}}(\Gamma) &= \Big\{ (q_i) \in \Pi_{i=0}^{n} \mathrm{H}^{-\frac{1}{2}}(\partial\Omega_i) \;\Big|\; \exists \mathbf{q} \in \mathrm{H}(\mathrm{div},\Omega) \text{ with } n_j\cdot\mathbf{q}|_{\partial\Omega_j} = q_j,\ \forall j = 0\dots n \Big\}\\
\mathbb{X}(\Gamma) &= \Big\{ (v_j, q_j)_{0\le j\le n} \in \mathbb{H}(\Gamma) \;\Big|\; (v_j) \in \mathbb{X}^{+\frac{1}{2}}(\Gamma) \text{ and } (q_j) \in \mathbb{X}^{-\frac{1}{2}}(\Gamma) \Big\}\\
\hat{\mathbb{X}}^{+\frac{1}{2}}(\Gamma) &= \Big\{ (v_i) \in \Pi_{i=0}^{n+1} \mathrm{H}^{\frac{1}{2}}(\partial\Omega_i) \;\Big|\; \exists v \in \mathrm{H}^1_{\mathrm{loc}}(\Omega) \text{ with } v|_{\partial\Omega_j} = v_j,\ \forall j = 0\dots n \text{ and } \gamma^{n+1}_{\mathrm{D,c}}(v) = v_{n+1} \Big\}\\
\hat{\mathbb{X}}^{-\frac{1}{2}}(\Gamma) &= \Big\{ (q_i) \in \Pi_{i=0}^{n+1} \mathrm{H}^{-\frac{1}{2}}(\partial\Omega_i) \;\Big|\; \exists \mathbf{q} \in \mathrm{H}(\mathrm{div},\Omega) \text{ with } n_j\cdot\mathbf{q}|_{\partial\Omega_j} = q_j,\ \forall j = 0\dots n+1 \Big\}\\
\hat{\mathbb{X}}(\Gamma) &= \Big\{ (v_j, q_j)_{0\le j\le n+1} \in \mathbb{H}(\Gamma) \;\Big|\; (v_j) \in \hat{\mathbb{X}}^{+\frac{1}{2}}(\Gamma) \text{ and } (q_j) \in \hat{\mathbb{X}}^{-\frac{1}{2}}(\Gamma) \Big\}\\
\mathbb{X}^{+\frac{1}{2}}_{0,\mathrm{D}}(\Gamma) &= \Big\{ (v_i) \in \Pi_{i=0}^{n} \mathrm{H}^{\frac{1}{2}}(\partial\Omega_i) \;\Big|\; \exists v \in \mathrm{H}^1_{0,\mathrm{loc}}(\Omega) \text{ with } v|_{\partial\Omega_j} = v_j,\ \forall j = 0\dots n \Big\}\\
\mathbb{X}_{0,\mathrm{D}}(\Gamma) &= \Big\{ (v_j, q_j)_{0\le j\le n} \in \mathbb{H}(\Gamma) \;\Big|\; (v_j) \in \mathbb{X}^{+\frac{1}{2}}_{0,\mathrm{D}}(\Gamma) \text{ and } (q_j) \in \mathbb{X}^{-\frac{1}{2}}(\Gamma) \Big\}
\end{aligned}
$$

To get an intuition of these spaces, observe that in the case where $\mathbb{R}^d = \overline{\Omega}_0 \cup \overline{\Omega}_1$ so that $\Gamma = \partial\Omega_0 = \partial\Omega_1$, there holds $\mathbb{X}(\Gamma) = \{\ (v, q, v, -q) \mid v \in \mathrm{H}^{1/2}(\Gamma),\ q \in \mathrm{H}^{-1/2}(\Gamma)\ \}$.

Let us prove a result of duality for $\mathbb{X}_{0,\mathrm{D}}(\Gamma)$. This result is well known for $\mathbb{X}(\Gamma)$ and $\hat{\mathbb{X}}(\Gamma)$.

Proposition 2.1.
Let $(u_j) \in \Pi_{j=0}^{n}\mathrm{H}^{+\frac{1}{2}}(\partial\Omega_j)$ and $(p_j) \in \Pi_{j=0}^{n}\mathrm{H}^{-\frac{1}{2}}(\partial\Omega_j)$. We have

$$
\begin{aligned}
&\text{(i)} \quad (u_j) \in \mathbb{X}^{+\frac{1}{2}}_{0,\mathrm{D}}(\Gamma) \iff \sum_{j=0}^{n} \int_{\partial\Omega_j} u_j q_j \, d\sigma = 0 \quad \forall (q_j) \in \mathbb{X}^{-\frac{1}{2}}(\Gamma)\\
&\text{(ii)} \quad (p_j) \in \mathbb{X}^{-\frac{1}{2}}(\Gamma) \iff \sum_{j=0}^{n} \int_{\partial\Omega_j} p_j v_j \, d\sigma = 0 \quad \forall (v_j) \in \mathbb{X}^{+\frac{1}{2}}_{0,\mathrm{D}}(\Gamma)
\end{aligned}
$$

Proof:
We only present the proof for (i) since the proof for (ii) is very similar.
$\boxed{\Rightarrow}$ First assume that $(u_j) \in \mathbb{X}^{1/2}_{0,\mathrm{D}}(\Gamma)$: $\exists u \in \mathrm{H}^1_{0,\mathrm{loc}}(\Omega)$ such that $u|_{\partial\Omega_j} = u_j$, $j = 0\dots n$. Since Γ is bounded, we may assume that $\mathrm{supp}(u)$ is bounded, using a cut-off function if necessary : so we can take $u \in \mathrm{H}^1_0(\Omega)$. Let us define $\hat{u} \in \mathrm{H}(\mathbb{R}^d)$ the extension of u on all $\mathbb{R}^d$ by

$$
\hat{u} = \begin{cases} u & \text{in } \Omega \\ \tilde{u} \in \mathrm{H}^1(\Omega_{n+1}) & \text{in } \Omega_{n+1} \text{ with } \tilde{u}|_{\partial\Omega_{n+1}} = u|_{\partial\Omega_{n+1}} = 0 \end{cases}
$$

Then we can define $u_{n+1} = u|_{\partial\Omega_{n+1}} = 0$. Consider an arbitary $(q_j) \in \mathbb{X}^{-1/2}(\Gamma)$: $\exists \mathbf{q} \in \mathrm{H}(\mathrm{div},\Omega)$ such that $n_j \cdot \mathbf{q}|_{\partial\Omega_j} = q_j$. We can extend $\mathbf{q}$ on all $\mathbb{R}^d$ as well and the extension $\hat{\mathbf{q}}$ is defined by

$$
\hat{\mathbf{q}} = \begin{cases} \mathbf{q} & \text{in } \Omega \\ \tilde{\mathbf{q}} \in \mathrm{H}(\mathrm{div},\Omega_{n+1}) & \text{in } \Omega_{n+1} \text{ with } n_{n+1}\cdot\tilde{\mathbf{q}}|_{\partial\Omega_{n+1}} = n_{n+1}\cdot\mathbf{q}|_{\partial\Omega_{n+1}} \end{cases}
$$

Then we can define $q_{n+1} = n_{n+1} \cdot \mathbf{q}|_{\partial\Omega_{n+1}}$. We obtain applying Green's Formula

$$\begin{aligned}\sum_{j=0}^{n} \int_{\partial\Omega_j} u_j q_j d\sigma = \sum_{j=0}^{n} \int_{\partial\Omega_j} u_j q_j d\sigma + \underbrace{\int_{\partial\Omega} u_{n+1} q_{n+1} d\sigma}_{=0} &= \sum_{j=0}^{n+1} \int_{\partial\Omega_j} u_j q_j d\sigma \\ &= \sum_{j=0}^{n+1} \int_{\Omega_j} \hat{u}\,\mathrm{div}(\hat{\mathbf{q}}) + \nabla\hat{u}\cdot\hat{\mathbf{q}}\; d\mathbf{x} \\ &= \int_{\mathbb{R}^d} \hat{u}\,\mathrm{div}(\hat{\mathbf{q}}) + \nabla\hat{u}\cdot\hat{\mathbf{q}}\; d\mathbf{x} = 0\end{aligned}$$

$\boxed{\Leftarrow}$ Now let us consider an arbitrary $(u_j) \in \Pi_{j=0}^{n} \mathrm{H}^{+\frac{1}{2}}(\partial\Omega_j)$ and assume that it satisfies the condition in the right hand side of (i). For any $j = 0 \ldots n$ there exists $v_j \in \mathrm{H}^1(\Omega_j)$ such that $v_j|_{\partial\Omega_j} = u_j$. Define $u \in \mathrm{L}^2(\Omega)$ by $u|_{\Omega_j} = v_j$, and $\mathbf{p} \in \mathrm{L}^2(\Omega)^3$ by $\mathbf{p}|_{\Omega_j} = \nabla v_j$. For any $\mathbf{q} \in \mathrm{H}_0(\mathrm{div}, \Omega)$, setting $q_j = n_j \cdot \mathbf{q}|_{\partial\Omega_j}$, we have

$$\begin{aligned}\int_{\Omega} u\,\mathrm{div}(\mathbf{q})\, d\mathbf{x} = \sum_{j=0}^{n} \int_{\Omega_j} v_j\,\mathrm{div}(\mathbf{q})\, d\mathbf{x} &= \underbrace{\sum_{j=0}^{n} \int_{\partial\Omega_j} u_j\, q_j\, d\sigma}_{=0} - \sum_{j=0}^{n} \int_{\Omega_j} \nabla v_j \cdot \mathbf{q}\, d\mathbf{x} \\ &= -\int_{\Omega} \mathbf{p}\cdot\mathbf{q}\, d\mathbf{x}\end{aligned}$$

as $(q_j) \in \mathbb{X}^{-\frac{1}{2}}(\Gamma)$ by definition. Since the preceding identity holds for any $\mathbf{q} \in \mathrm{H}_0(\mathrm{div}, \Omega)$, this proves that $u \in \mathrm{H}^1(\Omega)$. We have now to prove that $u|_{\partial\Omega} = 0$.
Now let us take $\mathbf{q} \in \mathrm{H}(\mathrm{div}, \Omega)$ so $n_{\partial\Omega} \cdot \mathbf{q}|_{\partial\Omega} \in \mathrm{H}^{-\frac{1}{2}}(\partial\Omega)$ is well defined. And then applying Green's Formula and considering the result we have obtained (which is true for $\mathbf{q} \in \mathrm{H}(\mathrm{div}, \Omega)$ as well)

$$\int_{\Omega} u\,\mathrm{div}(\mathbf{q})\, d\mathbf{x} = \int_{\partial\Omega} u|_{\partial\Omega}\; n_{\partial\Omega}\cdot\mathbf{q}|_{\partial\Omega} d\sigma - \int_{\Omega} \mathbf{p}\cdot\mathbf{q}\, d\mathbf{x} = -\int_{\Omega} \mathbf{p}\cdot\mathbf{q}\, d\mathbf{x}$$

which proves that :

$$\int_{\partial\Omega} \underbrace{u|_{\partial\Omega}}_{\in \mathrm{H}^{\frac{1}{2}}(\partial\Omega)} \;\underbrace{n_{\partial\Omega}\cdot\mathbf{q}|_{\partial\Omega}}_{\in \mathrm{H}^{-\frac{1}{2}}(\partial\Omega)} d\sigma = 0, \quad \forall \mathbf{q} \in \mathrm{H}(\mathrm{div}, \Omega)$$

so that $u|_{\partial\Omega} = 0$ because $\mathbf{q} \in \mathrm{H}(\mathrm{div}, \Omega) \mapsto n_{\partial\Omega} \cdot \mathbf{q}|_{\partial\Omega} \in \mathrm{H}^{-\frac{1}{2}}(\partial\Omega)$ is surjective. $\square$

Moreover, one obvious consequence of the preceding proposition is that $\mathbb{X}_{0,\mathrm{D}}(\Gamma)$ can be identified with its own polar set under the duality pairing $\mathrm{B}(\,,\,)$. More precisely : for any $U \in \mathbb{H}(\Gamma)$ we have

$$U \in \mathbb{X}_{0,\mathrm{D}}(\Gamma) \quad \Longleftrightarrow \quad \mathrm{B}(U, V) = 0 \quad \forall V \in \mathbb{X}_{0,\mathrm{D}}(\Gamma) \tag{4}$$

Let us recall that this is also true for U and V in $\mathbb{X}(\Gamma)$ and in $\hat{\mathbb{X}}(\Gamma)$.

Let us prove now a result of uniqueness in $\mathbb{X}_{0,\mathrm{D}}(\Gamma)$ which is usefull to obtain formulations of first kind like von Petersdorff obtained in [7]. We won't use this result because we will only derivate formulations of second kind but it is a quite interesting result.
The operator C_κ refers to the operator defined in [3].

Proposition 2.2.
For any $F \in \mathbb{H}(\Gamma)$, there exists a unique $U \in \mathbb{X}_{0,\mathrm{D}}(\Gamma)$ such that

$$\mathrm{B}(\mathrm{C}_\kappa U, V) = \mathrm{B}(F, V) \quad \forall V \in \mathbb{X}_{0,\mathrm{D}}(\Gamma)\ .$$

Proof:
According to Fredholm alternative, in order to prove the result, we only need to show that the only $U \in \mathbb{X}_{0,\mathrm{D}}(\Gamma)$ satisfying $\mathrm{B}(\mathrm{C}_\kappa U, V) = 0$, $\forall V \in \mathbb{X}_{0,\mathrm{D}}(\Gamma)$ is $U = 0$.
Let us take any $U = (U_0, \dots, U_n)^\top \in \mathbb{X}_{0,\mathrm{D}}(\Gamma)$ satisfying $\mathrm{B}(\mathrm{C}_\kappa U, V) = 0$, $\forall V \in \mathbb{X}_{0,\mathrm{D}}(\Gamma)$. Let us define $\psi_j(\mathbf{x}) = \mathrm{G}^j_{\kappa_j}\{U_j\}(\mathbf{x})$ so : $U_j = [\gamma^j] \cdot G^j_{\kappa_j}(U_j) = \gamma^j(\psi_j) - \gamma^j_c(\psi_j)$.

First, let us prove that $\psi_j = 0$ in Ω_j for all $j = 0 \dots n$. Lets us consider $\varphi \in \mathrm{L}^2_{\mathrm{loc}}(\Omega)$ such that $\varphi|_{\Omega_j} = \psi_j$, and set $W_{\mathrm{int}} = (\mathrm{Id}/2 + \mathrm{C}_\kappa)U$. We have $W_{\mathrm{int}} = (\gamma^0(\varphi), \dots, \gamma^n(\varphi))$ and since $\mathrm{B}(W_{\mathrm{int}}, V) = \mathrm{B}((\mathrm{Id}/2 + \mathrm{C}_\kappa)U, V) = \mathrm{B}(\mathrm{C}_\kappa U, V) = 0\,, \forall V \in \mathbb{X}_{0,\mathrm{D}}(\Gamma)$, we deduce that $W_{\mathrm{int}} \in \mathbb{X}_{0,\mathrm{D}}(\Gamma)$ according to (4), hence $\varphi \in \mathrm{H}^1_{0,\mathrm{loc}}(\Delta, \Omega)$. To sum up

$$\begin{aligned}
&\varphi \in \mathrm{H}^1_{0,\mathrm{loc}}(\Delta, \Omega) \quad \text{such that}\\
&\Delta\varphi + \kappa_j^2 \varphi = 0 \quad \text{in } \Omega_j,\ j = 0 \dots n\\
&\varphi \text{ is outgoing radiating in } \Omega_0.
\end{aligned}$$

As a consequence φ is solution to an homogeneous transmission problem (that is classically well posed). Hence $\varphi = 0$ i.e. $\psi_j = 0$ in Ω_j for all $j = 0 \dots n$.

Now let us show that $\psi_j = 0$ in $\Omega \setminus \overline{\Omega}_j$ for all $j = 0 \dots n$. Set $W_{\mathrm{ext}} = -(\mathrm{Id}/2 - \mathrm{C}_\kappa)U$. We have $W_{\mathrm{ext}} = (\gamma^0_c(\psi_0), \dots, \gamma^n_c(\psi_n))$ and $\mathrm{B}(W_{\mathrm{ext}}, V) = -\mathrm{B}((\mathrm{Id}/2 - \mathrm{C}_\kappa)U, V) = \mathrm{B}(\mathrm{C}_\kappa U, V) = 0\,, \forall V \in \mathbb{X}_{0,\mathrm{D}}(\Gamma)$ so $W_{\mathrm{ext}} \in \mathbb{X}_{0,\mathrm{D}}(\Gamma)$ according to (4). Clearly

$$\begin{aligned}
&\Delta\psi_j + \kappa_j^2 \psi_j = 0 \quad \text{in } \Omega \setminus \overline{\Omega}_j,\ j = 0 \dots n\\
&\psi_j \text{ is outgoing for } j \neq 0.
\end{aligned}$$

Let us consider $v = \overline{\Psi}_j$ and $\mathbf{q} = \nabla(\Psi_j)$, we have $v \in \mathrm{H}^1_{0,\mathrm{loc}}(\Omega)$ and $\mathbf{q} \in \mathrm{H}_{\mathrm{loc}}(\mathrm{div}, \Omega)$ since $W_{\mathrm{ext}} \in \mathbb{X}_{0,\mathrm{D}}(\Gamma)$. We can extend v and $\mathbf{q}$ on all $\mathbb{R}^d$ like we did in Proposition 2.1.
Take $r > 0$ large enough to ensure that $(\mathbb{R}^d \setminus \Omega_0) \subset \mathrm{B}_r = \{\, \mathbf{x} \in \mathbb{R}^d \mid |\mathbf{x}| < r \,\}$. Then we have :

$$\begin{aligned}
&\int_{\mathrm{B}_r \setminus (\overline{\Omega}_j \cup \overline{\Omega}_{n+1})} |\nabla\psi_j|^2 - \kappa_j^2 \,|\psi_j|^2 \, d\mathbf{x}\\
= \ &\int_{\mathrm{B}_r \setminus (\overline{\Omega}_j \cup \overline{\Omega}_{n+1})} \mathbf{q} \cdot \nabla v + v \,\mathrm{div}(\mathbf{q}) \, d\mathbf{x}\\
= \ &\int_{\mathrm{B}_r} \mathbf{q} \cdot \nabla v + v \,\mathrm{div}(\mathbf{q}) \, d\mathbf{x} - \int_{\Omega_j} \mathbf{q} \cdot \nabla v + v \,\mathrm{div}(\mathbf{q}) \, d\mathbf{x} - \int_{\Omega_{n+1}} \mathbf{q} \cdot \nabla v + v \,\mathrm{div}(\mathbf{q}) \, d\mathbf{x}\\
= \ &\int_{\partial\mathrm{B}_r} \overline{\psi}_j \,\partial_r \psi_j d\sigma - \int_{\partial\Omega_j} \overline{\psi}_j \,\partial_{n_j} \psi_j d\sigma - \underbrace{\int_{\partial\Omega_{n+1}} v|_{\partial\Omega_{n+1}} \,\mathbf{q}|_{\partial\Omega_{n+1}} \cdot n_{n+1} d\sigma}_{=0}
\end{aligned}$$

Finally we have :

$$\int_{\partial \mathrm{B}_r} \psi_j\, \partial_r \overline{\psi}_j d\sigma \;=\; \int_{\mathrm{B}_r \setminus (\overline{\Omega}_j \cup \overline{\Omega}_{n+1})} |\nabla \psi_j|^2 - \kappa_j^2\, |\psi_j|^2\, d\mathbf{x} + \int_{\partial\Omega_j} \gamma^j_{\mathrm{D},c}(\psi_j)\gamma^j_{\mathrm{N},c}(\overline{\psi}_j) d\sigma \quad \forall j \neq 0$$

$$0 \;=\; \int_{\mathrm{B}_r \setminus (\overline{\Omega}_0 \cup \overline{\Omega}_{n+1})} |\nabla \psi_0|^2 - \kappa_0^2\, |\psi_0|^2\, d\mathbf{x} + \int_{\partial\Omega_0} \gamma^0_{\mathrm{D},c}(\psi_0)\gamma^0_{\mathrm{N},c}(\overline{\psi}_0) d\sigma.$$

Take the imaginary part of the identity above, and sum over $j = 0 \dots n$, taking into account that $(\gamma^j_{\mathrm{D},c}(\psi_j))_{0\leq j\leq n} \in \mathbb{X}^{1/2}(\Gamma)$ and $(\gamma^j_{\mathrm{N},c}(\psi_j))_{0\leq j\leq n} \in \mathbb{X}^{-1/2}_{0,\mathrm{D}}(\Gamma)$ (since $W_{\mathrm{ext}} \in \mathbb{X}_{0,\mathrm{D}}(\Gamma)$). This yields

$$\sum_{j=1}^{n} \mathrm{Im}\Big\{ \int_{\partial \mathrm{B}_r} \psi_j\, \partial_n \overline{\psi}_j d\sigma \Big\} \;=\; \mathrm{Im}\Big\{ \sum_{j=0}^{n} \int_{\partial\Omega_j} \gamma_{\mathrm{D},c}(\psi_j)\gamma_{\mathrm{N},c}(\overline{\psi}_j)\, d\sigma \Big\} = 0.$$

In the last equality above we used Proposition 2.1. Note that, by construction, ψ_j is outgoing radiating with respect to the wave number κ_j. Combining this condition at infinity with the identity above for $j = 1 \dots n$ yields

$$\sum_{j=1}^{n} \int_{\partial \mathrm{B}_r} |\partial_r \psi_j|^2 + \kappa_j^2 |\psi_j|^2 d\sigma$$
$$= \sum_{j=1}^{n} \int_{\partial \mathrm{B}_r} |\partial_r \psi_j - i\kappa_j \psi_j|^2 d\sigma - \sum_{j=1}^{n} \mathrm{Im}\Big\{ \int_{\partial \mathrm{B}_r} \psi_j\, \partial_r \overline{\psi}_j d\sigma \Big\} \underset{r\to\infty}{\longrightarrow} 0.$$

This shows in particular that $\lim_{r\to\infty} \int_{\partial \mathrm{B}_r} |\psi_j|^2 d\sigma = 0$ for all $j = 1 \dots n$. As a consequence, we can apply Rellich Lemma, see Lemma 2.11 in [4], which implies that $\psi_j = 0$ in $\Omega \setminus \overline{\Omega}_j, j = 1 \dots n$. There only remains to deal with ψ_0. According to the transmission conditions satisfied by ψ_0 we have $\gamma^0_{\mathrm{D},c}(\psi_0) = 0$ and $\gamma^0_{\mathrm{N},c}(\psi_0) = 0$. Hence $-\psi_0(\mathbf{x}) = \mathrm{G}^0_{\kappa_0}\{\gamma^0_c(\psi_0)\}(\mathbf{x}) = 0$ in $\Omega \setminus \overline{\Omega}_0$. Since $U_j = \gamma^j(\psi_j) - \gamma^j_c(\psi_j) = 0$ for all $j = 0 \dots n$, $U = 0$. □

3 Single trace formulation of second kind for scattering by one isolated metallic domain

In this section we study the acoustic scattering by one isolated metallic domain so we consider two domains : the exterior domain Ω_0 and the metallic domain Ω_1.

3.1 Theoretical study

We conisder a partition $\mathbb{R}^d = \overline{\Omega}_0 \cup \overline{\Omega}_1$ where Ω_1 is bounded. We set $\Gamma = \partial\Omega_0 = \partial\Omega_1$.

The problem that we study Let $u_{\mathrm{inc}} \in \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d)$ satisfy $\Delta u_{\mathrm{inc}} + \kappa^2\, u_{\mathrm{inc}} = 0$ in $\mathbb{R}^d$ for some $\kappa \in \mathbb{R}$. This function plays the role of incident field. We study the following problem:

$$\text{Find } \; u \in \mathrm{H}^1_{0,\mathrm{loc}}(\Delta, \Omega_0) \; \text{ such that} \tag{5}$$

$$\begin{cases} \Delta u + \kappa^2 u = 0 \quad \text{in } \; \Omega_0 \\ u - u_{\mathrm{inc}} \; \text{ outgoing radiating in } \Omega_0 \end{cases} \tag{6}$$

Adapted functionnal space To derivate the variational formulation we need to introduce a new trace space. This space is a complementary of $\mathbb{X}_{0,\mathrm{D}}(\Gamma)$ in $\mathbb{H}(\Gamma)$ and we call it $\mathbb{Y}_{0,\mathrm{N}}(\Gamma)$. This space is defined by

$$
\begin{aligned}
\mathbb{Y}_{0,\mathrm{N}}(\Gamma) \quad &= \Big\{ (v_j, q_j)_{0\leq j\leq n} \in \mathbb{H}(\Gamma) \ \Big| \ (v_j) \in \mathbb{X}^{+\frac{1}{2}}(\Gamma) \ \text{ and } \ (q_j) \in \mathbb{Y}^{-\frac{1}{2}}_{0,\mathrm{N}}(\Gamma) \Big\} \\
\text{with} \quad \mathbb{Y}^{-\frac{1}{2}}_{0,\mathrm{N}}(\Gamma) \quad &= \Big\{ (q_i) \in \Pi_{i=0}^{n} \mathrm{H}^{-\frac{1}{2}}(\partial\Omega_i) \ \Big| \ \exists \mathbf{q} \in \mathrm{H}_0(\mathrm{div}, \Omega) \ \text{ with } \ n_j \cdot \mathbf{q}|_{\partial\Omega_j} = q_j, \ \forall j = 0 \ldots n \Big\}
\end{aligned}
$$

Variatonal formulation Set $U_{\mathrm{inc}} = (\gamma^0_{\mathrm{D}}(u_{\mathrm{inc}}), \gamma^0_{\mathrm{N}}(u_{\mathrm{inc}})) = (v_{inc}, q_{inc}) \in \mathbb{X}(\Gamma)$ and $U = (\gamma^0_{\mathrm{D}}(u), \gamma^0_{\mathrm{N}}(u)) = (v, q) \in \mathbb{X}_{0,\mathrm{D}}(\Gamma)$. Then it is well known that the variational formulation of (5)-(6) is :

$$
\text{Find } \ U \in \mathbb{X}_{0,\mathrm{D}}(\Gamma) \quad \text{such that} \quad (\frac{\mathrm{Id}}{2} - \mathrm{C}_\kappa)(U - U_{\mathrm{inc}}) = 0 \tag{7}
$$

First it is clear that $U = (0, q)$ since $u|_\Gamma = 0$ and (Lemma 4.1) $G^0_\kappa(\gamma^0(u_{inc})) + G^1_\kappa(\gamma^1(u_{inc})) = 0$ wich implies $\gamma^1 \cdot G^0_\kappa(\gamma^0(u_{inc})) + \gamma^1 \cdot G^1_\kappa(\gamma^1(u_{inc})) = 0$.
Remember that$\gamma^1 \cdot G^1_\kappa(\gamma^1(u_{inc})) = \gamma^1(u_{inc})$ since $\gamma^1(u_{inc})$ is a Cauchy data in Ω_1 and $\gamma^0_c(u_{inc}) = Q\ \gamma^1(u_{inc})$ with

$$
Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

we obtain $\gamma^0_c \cdot G^0_\kappa(\gamma^0(u_{inc})) + \gamma^0_c(u_{inc}) = 0$.
With $\gamma^0_c \cdot G^0_\kappa = -\frac{\mathrm{Id}}{2} + \mathrm{C}_\kappa$ and $\gamma^0_c(u_{inc}) = \gamma^0(u_{inc})$ since $u_{inc} \in \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d)$, we obtain $(\frac{\mathrm{Id}}{2} - \mathrm{C}_\kappa)U_{\mathrm{inc}} = U_{\mathrm{inc}}$.
The variational formulation (7) can be finally rewritten

$$
\begin{aligned}
&\text{Find } \ U = (0, q) \in \mathbb{X}_{0,\mathrm{D}}(\Gamma) \quad \text{such that} \\
&\mathrm{B}((\tfrac{\mathrm{Id}}{2} - \mathrm{C}_\kappa)U, V) = \mathrm{B}(U_{inc}, V) \qquad \forall V = (w, 0) \in \mathbb{Y}_{0,\mathrm{N}}(\Gamma)
\end{aligned}
$$

wich means

$$
\begin{aligned}
&\text{Find } \ q \in \mathrm{H}^{-\frac{1}{2}}(\Gamma) \quad \text{such that} \\
&\int_\Gamma (\frac{\mathrm{Id}}{2} - \tilde{\mathrm{K}})q\ w\ d\sigma = \int_\Gamma q_{\mathrm{inc}} w\ d\sigma \qquad \forall w \in \mathrm{H}^{\frac{1}{2}}(\Gamma)
\end{aligned} \tag{8}
$$

It is clear that (5)-(6) implies (8) : if u is solution of (5)-(6) then $q = \gamma^0_N(u)$ is solution of (8).

Let us prove the other implication considering $q \in \mathrm{H}^{-\frac{1}{2}}(\Gamma)$ such that

$$
\int_\Gamma (\frac{\mathrm{Id}}{2} - \tilde{K})q\ w\ d\sigma = \int_\Gamma q_{\mathrm{inc}} w\ d\sigma \quad \forall w \in \mathrm{H}^{\frac{1}{2}}(\Gamma)
$$

and define $\Psi(\mathbf{x}) = \mathrm{SL}_\kappa\{q\}(\mathbf{x}) \quad \forall \mathbf{x} \in \mathbb{R}^d \setminus \Gamma$. Clearly :

$$
\begin{cases}
-\Delta\Psi - \kappa^2\Psi = 0 \quad \text{in } \ \Omega_1 \\
\gamma^0_{N,C}(\Psi) = -q_{\mathrm{inc}} \\
\Psi \ \text{ outgoing}
\end{cases}
$$

and then we have $\Psi = -u_{\text{inc}}$ in Ω_1 because this problem is well posed so $\gamma_D^0(\Psi) = -\gamma_D^0(u_{\text{inc}})$. Let us define :

$$u = \underbrace{\Psi}_{\in \mathrm{H}^1_{\text{loc}}(\Delta,\mathbb{R}^d)} + \underbrace{u_{\text{inc}}}_{\in \mathrm{H}^1_{\text{loc}}(\Delta,\mathbb{R}^d)} \in \mathrm{H}^1_{\text{loc}}(\Delta,\Omega)$$

This function is equal to 0 in Ω_1 by cunstruction and we have $-\Delta u_{\text{inc}} - \kappa^2 u_{\text{inc}} = 0 \quad \text{in} \ \ \Omega_0$ and $-\Delta\Psi - \kappa^2\Psi = 0 \quad \text{in} \ \ \Omega_0$ so we finally have :

$$\begin{cases} \Delta u + \kappa^2 u = 0 \quad \text{in} \ \ \Omega_0 \\ \gamma_D^0(u) = 0 \\ u - u_{\text{inc}} \ \text{outgoing radiating in } \Omega_0 \end{cases} \tag{9}$$

with $q = [\gamma_N^0(\Psi)] = [\gamma_N^0(u)] = \gamma_N^0(u)$. □

3.2 Numerical experiments

In this paragraph we present numerical results obtained by discretizing formulation (8), in two dimensions, using a Galerkin method we describe below. In this numerical experiment we work on a diffraction problem of a plane wave by a circular metallic object. The interest is that the analytic solution is well known so we can test the code easily. We use GMSH to generate a mesh of the circle, a C-library of integral equations (developed by Patrick Meury) to assembly operators and MATLAB to plot results and any usefull information. All the functions we add are written in C-language. We don't use any C-library of linear algebra.

Discretized formulation For discretization, we considered a paneling $\Gamma = \cup_{i=1}^{\mathrm{N}} \Gamma_{\mathrm{h}}^i$ of the unit circle where each Γ_{h}^i is a segment. Then we considered the space $\mathbb{V}_h^{+\frac{1}{2}}(\Gamma)$ with

$$\mathbb{V}_h^{+\frac{1}{2}}(\Gamma) = \{v_h \in C^0(\Gamma) \mid v_h|_{\Gamma_{\mathrm{h}}^i} \in \mathbb{P}_1 \ \text{ for } \ \Gamma_{\mathrm{h}}^i \subset \Gamma, \ i = 1 \dots \mathrm{N}\}$$

$\mathbb{V}_h^{+\frac{1}{2}}(\Gamma)$ is an approximation space for $\mathrm{H}^{-\frac{1}{2}}(\Gamma)$ and $\mathrm{H}^{+\frac{1}{2}}(\Gamma)$.

Setting $Q = \{q^i\}_{i=1\dots\mathrm{N}}$ and $Q_{\text{inc}} = \{q_{\text{inc}}^i\}_{i=1\dots\mathrm{N}}$, the discretized formulation of (8) is

$$\begin{aligned} &\text{Find } \ q \in \mathbb{V}_h^{+\frac{1}{2}}(\Gamma) \quad \text{such that} \\ &(\frac{\mathrm{M}}{2} - \tilde{\mathrm{K}})\, Q = \mathrm{M}\, Q_{\text{inc}} \end{aligned} \tag{10}$$

where M is the mass matrix of $\mathbb{P}_1$ functions. Because we use C-language, we need to separate real and imaginary parts. Then the formulation (10) becomes

$$\begin{aligned} &\text{Find } \ q \in \mathbb{V}_h^{+\frac{1}{2}}(\Gamma) \quad \text{such that} \\ &\begin{pmatrix} \frac{\mathrm{M}}{2} - \Re e(\tilde{\mathrm{K}}) & \Im m(\tilde{\mathrm{K}}) \\ -\Im m(\tilde{\mathrm{K}}) & \frac{\mathrm{M}}{2} - \Re e(\tilde{\mathrm{K}}) \end{pmatrix} \begin{pmatrix} \Re e(Q) \\ \Im m(Q) \end{pmatrix} = \begin{pmatrix} \mathrm{M}\ \Re e(Q_{\text{inc}}) \\ \mathrm{M}\ \Im m(Q_{\text{inc}}) \end{pmatrix} \end{aligned} \tag{11}$$

Mesh generation We use GMSH to create the mesh which modelises the mettalic domain. In **Fig.1** and **Fig.2** we represent the circular mesh with GMSH and MATLAB.

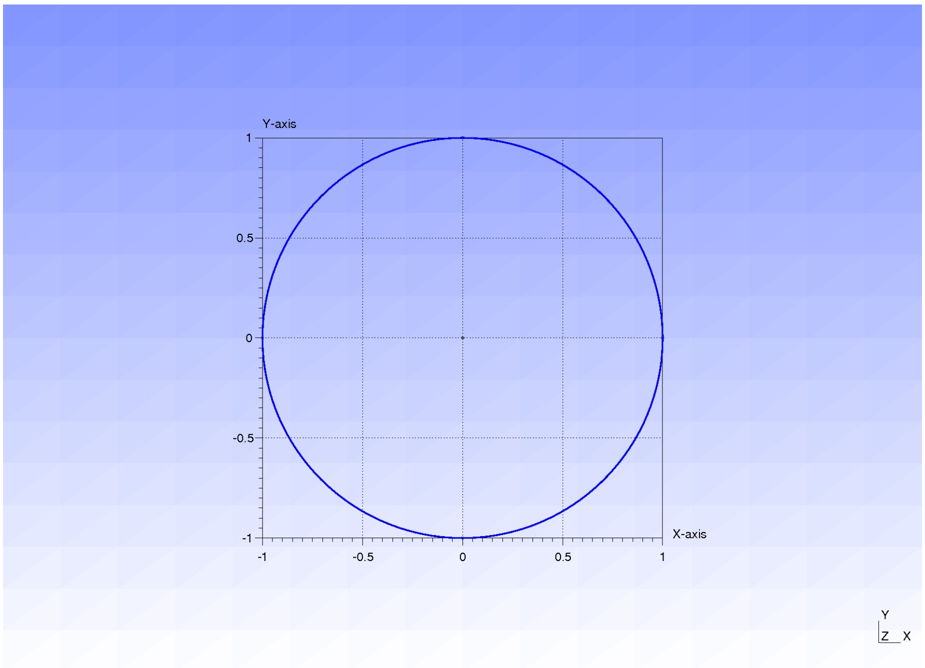

Fig.1 Unit circle displayed on GMSH

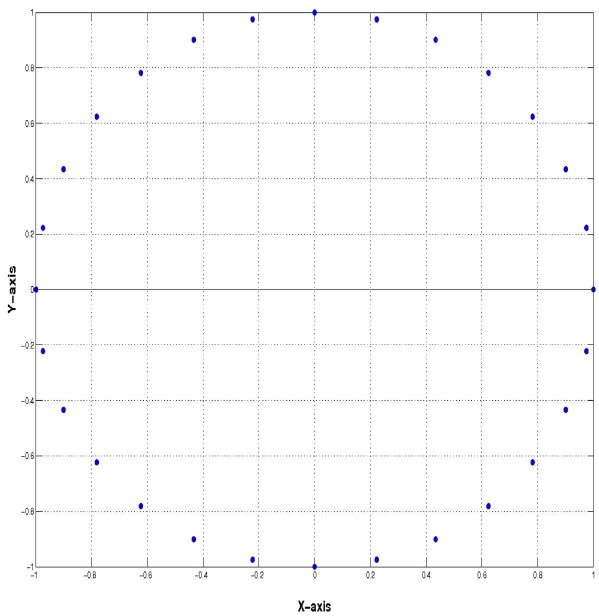

Fig.2 Circular Mesh ploted on Matlab with a step h = 0.2

The C-library of integral equations we use need a specific format of mesh so we have written a routine which gives this specific format from the .msh file generated by GMSH.

Assembly To assemble $\Re e(\mathrm{A})$ and $\Im m(\mathrm{A})$, we use the C-library of integral equations of Patrick Meury. Concerning the assembly of the load vector, we study the diffraction of a plave wave so we take $u_{\text{inc}}(r,\theta) = e^{i\kappa r\cos\theta}$ as an incident field. Using the Jacobi-Anger formula, we can write $u_{\text{inc}}(r,\theta) = e^{i\kappa r\cos\theta} = \sum_{n=-\infty}^{+\infty} i^n \mathrm{J}_n(\kappa r)e^{in\theta}$ and then

$$q_{\text{inc}}(r,\theta) = -\frac{\partial\, u_{\text{inc}}}{\partial r} = -\sum_{n=-\infty}^{+\infty} \kappa i^n \mathrm{J}'_n(\kappa r)e^{in\theta}$$

On the unit circle, q_{inc} is only a function of theta. Separating real and imaginary parts we obtain

$$\begin{aligned}
\Re e\left\{q_{\text{inc}}(\theta)\right\} &= \sum_{n=-\infty}^{+\infty} \frac{\kappa(-1)^n}{2}\Big(\mathrm{J}_{2n+1}(\kappa) - \mathrm{J}_{2n-1}(\kappa)\Big)\cos 2n.\theta \\
&+ \sum_{n=-\infty}^{+\infty} \frac{\kappa(-1)^{n+1}}{2}\Big(\mathrm{J}_{2n+2}(\kappa) - \mathrm{J}_{2n}(\kappa)\Big)\sin 2n+1.\theta
\end{aligned}$$

$$\begin{aligned}
\Im m\left\{q_{\text{inc}}(\theta)\right\} &= \sum_{n=-\infty}^{+\infty} \frac{\kappa(-1)^n}{2}\Big(\mathrm{J}_{2n+1}(\kappa) - \mathrm{J}_{2n-1}(\kappa)\Big)\sin 2n.\theta \\
&+ \sum_{n=-\infty}^{+\infty} \frac{\kappa(-1)^n}{2}\Big(\mathrm{J}_{2n+2}(\kappa) - \mathrm{J}_{2n}(\kappa)\Big)\cos 2n+1.\theta
\end{aligned}$$

Solving of the linear system To solve the linear system (11), we use the mininmum residual method. We have implemented this algortihm in C-language.

Test To test our code we compare the solution we obtain and the analytical solution which is given by the derivate $q(r,\theta)$ of $u(r,\theta) = u_{\text{inc}}(r,\theta) + u_{\text{diff}}(r,\theta)$ with

$$u_{\text{inc}}(r,\theta) = \sum_{n=-\infty}^{+\infty} i^n \mathrm{J}_n(\kappa r)e^{in\theta} \qquad \text{and} \qquad u_{\text{diff}}(r,\theta) = \sum_{n=-\infty}^{+\infty} \alpha_n \frac{\mathrm{H}_n^{(1)}(\kappa r)}{\mathrm{H}_n^{(1)}(\kappa)}e^{in\theta}$$

α_n is chosen to have $u(r=1,\theta) = 0,\ \ \forall\theta$, so we finally have

$$u(r,\theta) = \sum_{n=-\infty}^{+\infty} \Big(i^n \mathrm{J}_n(\kappa r) - i^n \frac{\mathrm{J}_n(\kappa)}{\mathrm{H}_n^{(1)}(\kappa)}\mathrm{H}_n^{(1)}(\kappa r)\Big)e^{in\theta}$$

and then

$$q(r,\theta) = -\frac{\partial\, u(r,\theta)}{\partial r} = \sum_{n=-\infty}^{+\infty} \kappa\Big(-i^n \mathrm{J}'_n(\kappa r) + i^n \frac{\mathrm{J}_n(\kappa)}{\mathrm{H}_n^{(1)}(\kappa)}\mathrm{H}_n^{(1)\prime}(\kappa r)\Big)e^{in\theta}$$

Using

$$\frac{\mathrm{H}_n^{(1)\prime}(\kappa r)}{\mathrm{H}_n^{(1)}(\kappa)} = \frac{\mathrm{H}_n^{(1)\prime}(\kappa r)\overline{\mathrm{H}_n^{(1)}}(\kappa)}{|\mathrm{H}_n^{(1)}(\kappa)|^2} = \frac{\Big(\mathrm{J}'_n(\kappa)\mathrm{J}_n(\kappa) + \mathrm{Y}'_n(\kappa)\mathrm{Y}_n(\kappa)\Big)}{|\mathrm{H}_n^{(1)}(\kappa)|^2} + i\frac{2}{\pi\kappa|\mathrm{H}_n^{(1)}(\kappa)|^2} \quad \text{for} \quad r=1$$

and writting $q(r,\theta) = q^1(r,\theta) + q^2(r,\theta) + q^3(r,\theta)$ we obtain

$$q^1(\theta) = \sum_{n=-\infty}^{+\infty} \frac{\kappa \mathrm{J}_n(\kappa) i^n}{2|\mathrm{H}_n^{(1)}(\kappa)|^2} \Big([\mathrm{J}_{n-1}(\kappa) - \mathrm{J}_{n+1}(\kappa)]\mathrm{J}_n(\kappa) + [\mathrm{Y}_{n-1}(\kappa) - \mathrm{Y}_{n+1}(\kappa)]\mathrm{Y}_n(\kappa) \Big) e^{in\theta}$$

$$q^2(\theta) = \sum_{n=-\infty}^{+\infty} \frac{2\mathrm{J}_n(\kappa) i^{n+1}}{\pi|\mathrm{H}_n^{(1)}(\kappa)|^2} e^{in\theta}$$

$$q^3(\theta) = \sum_{n=-\infty}^{+\infty} \frac{\kappa i^n}{2} \Big(\mathrm{J}_{n+1}(\kappa) - \mathrm{J}_{n-1}(\kappa) \Big) e^{in\theta}$$

$$\begin{aligned}
\Re e \left\{ q^1(\theta) \right\} = \quad & \sum_{n=-\infty}^{+\infty} \frac{\kappa(-1)^n}{2} \frac{\mathrm{J}_{2n}(\kappa)}{|\mathrm{H}_{2n}^{(1)}(\kappa)|^2} \Big([\mathrm{J}_{2n-1}(\kappa) - \mathrm{J}_{2n+1}(\kappa)]\mathrm{J}_{2n}(\kappa) \Big) \cos 2n.\theta \\
& + \sum_{n=-\infty}^{+\infty} \frac{\kappa(-1)^n}{2} \frac{\mathrm{J}_{2n}(\kappa)}{|\mathrm{H}_{2n}^{(1)}(\kappa)|^2} \Big([\mathrm{Y}_{2n-1}(\kappa) - \mathrm{Y}_{2n+1}(\kappa)]\mathrm{Y}_{2n}(\kappa) \Big) \cos 2n.\theta \\
& + \sum_{n=-\infty}^{+\infty} \frac{\kappa(-1)^{n+1}}{2} \frac{\mathrm{J}_{2n+1}(\kappa)}{|\mathrm{H}_{2n+1}^{(1)}(\kappa)|^2} \Big([\mathrm{J}_{2n}(\kappa) - \mathrm{J}_{2n+2}(\kappa)]\mathrm{J}_{2n+1}(\kappa) \Big) \sin 2n+1.\theta \\
& + \sum_{n=-\infty}^{+\infty} \frac{\kappa(-1)^{n+1}}{2} \frac{\mathrm{J}_{2n+1}(\kappa)}{|\mathrm{H}_{2n+1}^{(1)}(\kappa)|^2} \Big([\mathrm{Y}_{2n}(\kappa) - \mathrm{Y}_{2n+2}(\kappa)]\mathrm{Y}_{2n+2}(\kappa) \Big) \sin 2n+1.\theta
\end{aligned}$$

$$\begin{aligned}
\Im m \left\{ q^1(\theta) \right\} = \quad & \sum_{n=-\infty}^{+\infty} \frac{\kappa(-1)^n}{2} \frac{\mathrm{J}_{2n}(\kappa)}{|\mathrm{H}_{2n}^{(1)}(\kappa)|^2} \Big([\mathrm{J}_{2n-1}(\kappa) - \mathrm{J}_{2n+1}(\kappa)]\mathrm{J}_{2n}(\kappa) \Big) \sin 2n.\theta \\
& + \sum_{n=-\infty}^{+\infty} \frac{\kappa(-1)^n}{2} \frac{\mathrm{J}_{2n}(\kappa)}{|\mathrm{H}_{2n}^{(1)}(\kappa)|^2} \Big([\mathrm{Y}_{2n-1}(\kappa) - \mathrm{Y}_{2n+1}(\kappa)]\mathrm{Y}_{2n}(\kappa) \Big) \sin 2n.\theta \\
& + \sum_{n=-\infty}^{+\infty} \frac{\kappa(-1)^n}{2} \frac{\mathrm{J}_{2n+1}(\kappa)}{|\mathrm{H}_{2n+1}^{(1)}(\kappa)|^2} \Big([\mathrm{J}_{2n}(\kappa) - \mathrm{J}_{2n+2}(\kappa)]\mathrm{J}_{2n+1}(\kappa) \Big) \cos 2n+1.\theta \\
& + \sum_{n=-\infty}^{+\infty} \frac{\kappa(-1)^n}{2} \frac{\mathrm{J}_{2n+1}(\kappa)}{|\mathrm{H}_{2n+1}^{(1)}(\kappa)|^2} \Big([\mathrm{Y}_{2n}(\kappa) - \mathrm{Y}_{2n+2}(\kappa)]\mathrm{Y}_{2n+2}(\kappa) \Big) \cos 2n+1.\theta
\end{aligned}$$

$$\begin{aligned}
\Re e \left\{ q^2(\theta) \right\} = \quad & \sum_{n=-\infty}^{+\infty} \frac{2(-1)^{n+1}}{\pi} \frac{\mathrm{J}_{2n}(\kappa)}{|\mathrm{H}_{2n}^{(1)}(\kappa)|^2} \sin 2n.\theta \quad + \sum_{n=-\infty}^{+\infty} \frac{2(-1)^{n+1}}{\pi} \frac{\mathrm{J}_{2n+1}(\kappa)}{|\mathrm{H}_{2n+1}^{(1)}(\kappa)|^2} \cos 2n+1.\theta \\
\Im m \left\{ q^2(\theta) \right\} = \quad & \sum_{n=-\infty}^{+\infty} \frac{2(-1)^{n}}{\pi} \frac{\mathrm{J}_{2n}(\kappa)}{|\mathrm{H}_{2n}^{(1)}(\kappa)|^2} \cos 2n.\theta \quad + \sum_{n=-\infty}^{+\infty} \frac{2(-1)^{n+1}}{\pi} \frac{\mathrm{J}_{2n+1}(\kappa)}{|\mathrm{H}_{2n+1}^{(1)}(\kappa)|^2} \sin 2n+1.\theta \\
\Re e \left\{ q^3(\theta) \right\} = \quad & \sum_{n=-\infty}^{+\infty} \frac{\kappa(-1)^n}{2} \Big(\mathrm{J}_{2n+1}(\kappa) - \mathrm{J}_{2n-1}(\kappa) \Big) \cos 2n.\theta \\
& \quad + \sum_{n=-\infty}^{+\infty} \frac{\kappa(-1)^{n+1}}{2} \Big(\mathrm{J}_{2n+2}(\kappa) - \mathrm{J}_{2n}(\kappa) \Big) \sin 2n+1.\theta \\
\Im m \left\{ q^3(\theta) \right\} = \quad & \sum_{n=-\infty}^{+\infty} \frac{\kappa(-1)^n}{2} \Big(\mathrm{J}_{2n+1}(\kappa) - \mathrm{J}_{2n-1}(\kappa) \Big) \sin 2n.\theta \\
& \quad + \sum_{n=-\infty}^{+\infty} \frac{\kappa(-1)^{n}}{2} \Big(\mathrm{J}_{2n+2}(\kappa) - \mathrm{J}_{2n}(\kappa) \Big) \cos 2n+1.\theta
\end{aligned}$$

Results In this paragragh we show the results we have obtained with our discretized formulation (11). As above, q is the exact solution of our problem (components q_i) ; let us call $q^{\rm h}$ the approximate solution (components $q_i^{\rm h}$). In figures **Fig.3** and **Fig.4** we represent the quadratic errors about real part and imaginary part as functions of the step of the mesh, for a few values of κ (in reality we represent quadratic errors multiplied by $\mathrm{h}^{-\frac{1}{2}}$ to simulate $\mathrm{L}^2(\Gamma)$ errors).

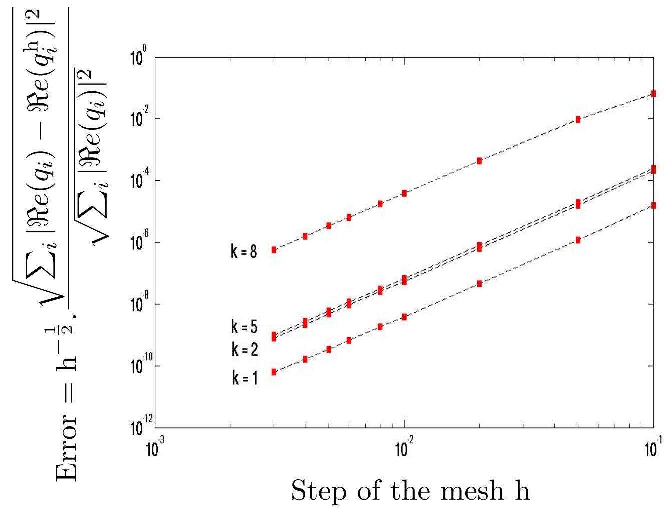

Fig.3 Error about real part for a few values of κ

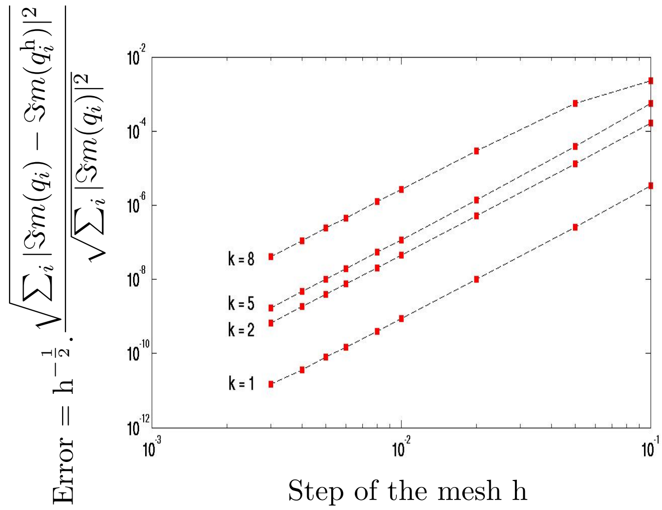

Fig.4 Error about imaginary part for a few values of κ

In figure **Fig.5** we represent the condition number of the matrix associated to (11) as a function of the step of the mesh, for $\kappa = 1$.

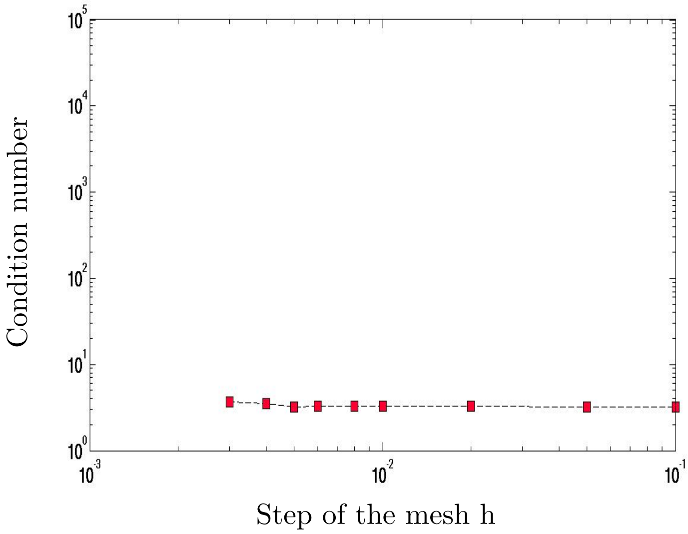

Fig.5 Stable conditioning

We observe that the condition number of this matrix is bounded independently of the step of the mesh : that is the principal interest of this method. The consequence is that the number of iterations to solve the linear system is independent of the step of the mesh.

4 Multi potentiel operator and single trace formulation of second kind for multiple subdomains scattering (including one metallic domain)

We study in this section the scattering of acoustic waves by an object composed of several adjacent parts with different material properties and one of the part is an impenetrable metallic domain. This is the heart of our report.

Let us define a continuous operator $\mathrm{A}_\kappa : \mathbb{H}(\Gamma) \to \mathbb{H}(\Gamma)$ by

$$\begin{aligned}
&\Phi(U)(\mathbf{x}) \;=\; \sum_{i=0}^{n} \mathrm{DL}^i_{\kappa_i}\{u_i\}(\mathbf{x}) + \mathrm{SL}^i_{\kappa_i}\{p_i\}(\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^d \setminus \Gamma \\
&\text{and} \quad \mathrm{A}_\kappa = (\gamma^0_{\mathrm{D}} \cdot \Phi, \gamma^0_{\mathrm{N}} \cdot \Phi, \dots, \gamma^n_{\mathrm{D}} \cdot \Phi, \gamma^n_{\mathrm{N}} \cdot \Phi)^\top \\
&\text{for} \quad U = (u_0, p_0, \dots, u_n, p_n)^\top
\end{aligned} \tag{12}$$

Important here is to note that, in Definition (12), all potentials are considered as functions defined *everywhere* except on Γ.

4.1 General case : one exterior domain Ω_0 and $n+1$ multiple subdomains

We consider a partition $\mathbb{R}^d = \cup_{i=0}^{n+1} \overline{\Omega}_i$ where $\overline{\cup_{i=1}^{n+1} \overline{\Omega}_i}$ is bounded and each Ω_i is a connnected open Lipschitz subset. We also set $\Omega = \mathbb{R}^d \setminus \overline{\Omega_{n+1}}$ ($\partial\Omega = \partial\Omega_{n+1}$) and $\Gamma = \cup_{i=0}^{n+1} \partial\Omega_i$.
Ω_{n+1} modelises an impenetrable metallic domain. We only study the case where all wave numbers are equals, then we have the lemma below.

Lemma 4.1.
Assume that $\kappa_j = \kappa, \forall j = 0 \dots n$. In this case, $\Phi(U)(\mathbf{x}) = 0 \ \forall \mathbf{x} \in \mathbb{R}^d$ for any $U \in \mathbb{X}(\Gamma)$ and for any $U \in \hat{\mathbb{X}}(\Gamma)$.

The problem that we study Let $u_{\mathrm{inc}} \in \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d)$ satisfy $\Delta u_{\mathrm{inc}} + \kappa^2 u_{\mathrm{inc}} = 0$ in $\mathbb{R}^d$ for some $\kappa \in \mathbb{R}$. This function plays the role of incident field. In the present report we study the following problem:

$$\text{Find} \;\; u \in \mathrm{H}^1_{0,\mathrm{loc}}(\Delta, \Omega) \;\; \text{such that} \tag{13}$$

$$\begin{cases} \Delta u + \kappa^2 u = 0 \quad \text{in} \;\; \Omega_i \;, \quad i = 0 \dots n \\ u - u_{\mathrm{inc}} \;\; \text{outgoing radiating in } \Omega_0 \end{cases} \tag{14}$$

Let us set $U_{\mathrm{inc}} = (\gamma^j(u_{\mathrm{inc}}))_{0 \le j \le n} \in \mathbb{X}(\Gamma)$ and $U = (U_j)_{0 \le j \le n} = (\gamma^j(u))_{0 \le j \le n} \in \mathbb{X}_{0,\mathrm{D}}(\Gamma)$. It is clear that if u is solution of (13)-(14) then $(\mathrm{Id} - A)(U - U_{\mathrm{inc}}) = 0$.

Let us define $F = (\mathrm{Id} - A)U_{\mathrm{inc}}$. Then our variational formulation of second kind of (13)-(14) is :

$$\begin{aligned}&\text{Find } \ U \in \mathbb{X}_{0,\mathrm{D}}(\Gamma) \quad \text{such that}\\ &\mathrm{B}((\mathrm{Id}-\mathrm{A})U, V) = \mathrm{B}(F, V) \qquad \forall V = \in \mathbb{Y}_{0,\mathrm{N}}(\Gamma)\end{aligned} \tag{15}$$

So if u is solution of (13)-(14) then $U = (\gamma^j(u))_{0\leq j\leq n}$ is solution of (15).
Let us prove the other implication considering $U \in \mathbb{X}_{0,\mathrm{D}}(\Gamma)$ wich verifies (15) and define $\Psi(\mathbf{x}) = \Phi(U)(\mathbf{x}) \ \ \forall \mathbf{x} \in \mathbb{R}^d \setminus \Gamma$: Ψ is outgoing and $-\Delta\Psi - \kappa^2\Psi = 0 \quad \text{in} \ \ \Omega_{n+1}$.
Let us prove that $\Psi = -u_{\mathrm{inc}} \in \Omega_{n+1}$: for that, we need to prove that $\gamma^{n+1}(\Psi) = -\gamma^{n+1}(u_{\mathrm{inc}})$. Then we will set $u = \Psi + u_{\mathrm{inc}} \in \mathrm{H}^1_{0,\mathrm{loc}}(\Delta, \Omega)$ with $\Delta u_{\mathrm{inc}} + \kappa^2 u_{\mathrm{inc}} = 0$ in Ω_i , $i = 0 \dots n$ and $\Delta\Psi + \kappa^2\Psi = 0$ in Ω_i , $i = 0 \dots n$ to have :

$$\left\{\begin{array}{l}\Delta u + \kappa^2 u = 0 \quad \text{in} \ \ \Omega_i \ , \quad i = 0 \dots n\\ u - u_{\mathrm{inc}} \ \ \text{outgoing radiating in } \Omega_0\end{array}\right.$$

Let us consider U_{n+1} and V_{n+1} in $\mathbb{H}(\partial\Omega)$ such that $(U, U_{n+1}) \in \hat{\mathbb{X}}(\Gamma)$ and $(V, V_{n+1}) \in \hat{\mathbb{X}}(\Gamma)$. Then it is clear that

$$\Psi(\mathbf{x}) = \Phi(U)(\mathbf{x}) = \underbrace{\sum_{i=0}^{n} \mathrm{G}^i_\kappa\{U_i\}(\mathbf{x}) + \mathrm{G}^{n+1}_\kappa\{U_{n+1}\}}_{=0} - \mathrm{G}^{n+1}_\kappa\{U_{n+1}\}$$

wich implies $U_{n+1} = -[\gamma^{n+1}(\Psi)]$, and

$$B(U, V) = \underbrace{\sum_{i=0}^{n} \mathrm{B}_i(U_i, V_i) + \mathrm{B}_{n+1}(U_{n+1}, V_{n+1})}_{=0} - \mathrm{B}_{n+1}(U_{n+1}, V_{n+1})$$

So finally $\mathrm{B}(U, V) = \mathrm{B}_{n+1}([\gamma^{n+1}(\Psi)], V_{n+1})$.
Moreover we have $AU = (\gamma^j \cdot \Phi(U))_{0\leq j\leq n} = (\gamma^j \cdot \Psi)_{0\leq j\leq n}$ and with the same calculation $\mathrm{B}(AU, V) = -\mathrm{B}_{n+1}(\gamma^{n+1}_c(\Psi), V_{n+1})$ so $\mathrm{B}((\mathrm{Id} - \mathrm{A})U, V) = \mathrm{B}_{n+1}(\gamma^{n+1}(\Psi), V_{n+1})$.
Now let us develop $\mathrm{B}((\mathrm{Id} - \mathrm{A})U_{\mathrm{inc}}, V)$. Since $(\gamma^j(u_{\mathrm{inc}}))_{0\leq j\leq n+1} \in \hat{\mathbb{X}}(\Gamma)$ we have

$$\sum_{i=0}^{n+1} \mathrm{G}^i_\kappa\{\gamma^i(u_{\mathrm{inc}})\}(\mathbf{x}) = 0$$

and then $\Phi(U_{\mathrm{inc}}) = -\mathrm{G}^{n+1}_\kappa\{\gamma^{n+1}(u_{\mathrm{inc}})\}$ and $\mathrm{A}U_{\mathrm{inc}} = (-\gamma^j \cdot \mathrm{G}^{n+1}_\kappa\{\gamma^{n+1}(u_{\mathrm{inc}})\})_{0\leq j\leq n} = 0$ because of the caracterisation of the integral representation. Then we have

$$\mathrm{B}((\mathrm{Id} - \mathrm{A})U_{\mathrm{inc}}, V) = \underbrace{\mathrm{B}(U_{\mathrm{inc}}, V)}_{=-\mathrm{B}_{n+1}(\gamma^{n+1}(u_{\mathrm{inc}}), V_{n+1})} - \underbrace{\mathrm{B}(\mathrm{A}U_{\mathrm{inc}}, V)}_{=0}$$

We finally obtain comparing $\mathrm{B}((\mathrm{Id} - \mathrm{A})U, V)$ and $\mathrm{B}((\mathrm{Id} - \mathrm{A})U_{\mathrm{inc}}, V)$

$$\mathrm{B}_{n+1}(\gamma^{n+1}(\Psi), V_{n+1}) = -\mathrm{B}_{n+1}(\gamma^{n+1}(u_{\mathrm{inc}}), V_{n+1}), \ \ \forall V_{n+1} \in \mathbb{H}(\partial\Omega)$$

which implies $\gamma^{n+1}(\Psi) = -\gamma^{n+1}(u_{\mathrm{inc}})$ because B is non-degenerate. $\square$

4.2 Toy problem : one exterior domain Ω_0 and two subdomains Ω_1 and Ω_2

We will explicit here our variational formulation (15) with a toy problem.

4.2.1 Theoretical study

We consider a partition $\mathbb{R}^2 = \overline{\Omega}_0 \cup \overline{\Omega}_1 \cup \overline{\Omega}_2$ where $\overline{\Omega}_1 \cup \overline{\Omega}_2$ is bounded and each Ω_i is a connnected open Lipschitz subset. Ω_0 and Ω_1 have the same physical properties so the boundary between theses two domains is fictional. Ω_2 modelises an impenetrable metallic domain : we work on a diffraction problem of a plane wave by a circular metallic object.
We set $\Omega = \mathbb{R}^2 \setminus \overline{\Omega_2}$, $\Gamma_0 = \partial\Omega_0$, $\Gamma_1 = \partial\Omega_1$, $\Gamma_2 = \partial\Omega_2$, $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$, $\Gamma_{01} = \Gamma_0 \cap \Gamma_1$, $\Gamma_{02} = \Gamma_0 \cap \Gamma_2$ and $\Gamma_{12} = \Gamma_1 \cap \Gamma_2$.

Trace spaces In this case $(n + 1 = 2)$ we have :

$$U = \begin{bmatrix} U_0(X) \\ U_1(Y) \end{bmatrix} = \begin{bmatrix} U_0^{\mathrm{D}}(X) \\ U_0^{\mathrm{N}}(X) \\ U_1^{\mathrm{D}}(Y) \\ U_1^{\mathrm{N}}(Y) \end{bmatrix} \in \mathbb{X}_{0,\mathrm{D}}(\Gamma) \iff \begin{cases} \begin{bmatrix} U_0^{\mathrm{D}}(X) \\ U_0^{\mathrm{N}}(X) \\ U_1^{\mathrm{D}}(Y) \\ U_1^{\mathrm{N}}(Y) \end{bmatrix} \in \mathrm{H}^{\frac{1}{2}}(\Gamma_0) \times \mathrm{H}^{-\frac{1}{2}}(\Gamma_0) \times \mathrm{H}^{\frac{1}{2}}(\Gamma_1) \times \mathrm{H}^{-\frac{1}{2}}(\Gamma_1) \\ \\ U_0^{\mathrm{D}}(X) = U_1^{\mathrm{D}}(X) \quad \forall X \in \Gamma_{01} \\ U_0^{\mathrm{D}}(X) = 0 \quad\quad\quad\; \forall X \in \Gamma_{02} \\ U_1^{\mathrm{D}}(Y) = 0 \quad\quad\quad\; \forall Y \in \Gamma_{12} \\ U_0^{\mathrm{N}}(Y) = -U_1^{\mathrm{N}}(Y) \quad \forall Y \in \Gamma_{01} \end{cases} \tag{16}$$

$$V = \begin{bmatrix} V_0(X) \\ V_1(Y) \end{bmatrix} = \begin{bmatrix} V_0^{\mathrm{D}}(X) \\ V_0^{\mathrm{N}}(X) \\ V_1^{\mathrm{D}}(Y) \\ V_1^{\mathrm{N}}(Y) \end{bmatrix} \in \mathbb{Y}_{0,\mathrm{N}}(\Gamma) \iff \begin{cases} \begin{bmatrix} V_0^{\mathrm{D}}(X) \\ V_0^{\mathrm{N}}(X) \\ V_1^{\mathrm{D}}(Y) \\ V_1^{\mathrm{N}}(Y) \end{bmatrix} \in \mathrm{H}^{\frac{1}{2}}(\Gamma_0) \times \mathrm{H}^{-\frac{1}{2}}(\Gamma_0) \times \mathrm{H}^{\frac{1}{2}}(\Gamma_1) \times \mathrm{H}^{-\frac{1}{2}}(\Gamma_1) \\ \\ V_0^{\mathrm{N}}(X) = V_1^{\mathrm{N}}(X) \quad \forall X \in \Gamma_{01} \\ V_0^{\mathrm{N}}(X) = 0 \quad\quad\quad\; \forall X \in \Gamma_{02} \\ V_1^{\mathrm{N}}(Y) = 0 \quad\quad\quad\; \forall Y \in \Gamma_{12} \\ V_0^{\mathrm{D}}(Y) = -V_1^{\mathrm{D}}(Y) \quad \forall Y \in \Gamma_{01} \end{cases} \tag{17}$$

The problem that we study Let $u_{\mathrm{inc}} \in \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d)$ satisfy $\Delta u_{\mathrm{inc}} + \kappa^2 u_{\mathrm{inc}} = 0$ in $\mathbb{R}^d$ for some $\kappa \in \mathbb{R}$. This function plays the role of incident field. In the present report we study the following problem:

$$\text{Find } \; u \in \mathrm{H}^1_{0,\mathrm{loc}}(\Delta, \Omega) \; \text{ such that} \tag{18}$$

$$\begin{cases} \Delta u + \kappa^2 u = 0 \quad \text{in } \; \Omega_0 \\ \Delta u + \kappa^2 u = 0 \quad \text{in } \; \Omega_1 \\ u - u_{\mathrm{inc}} \; \text{ outgoing radiating in } \Omega_0 \end{cases} \tag{19}$$

We have shown that (18)-(19) is equivalent to

$$\begin{aligned}&\text{Find } \ U \in \mathbb{X}_{0,\mathrm{D}}(\Gamma) \quad \text{such that}\\ &\mathrm{B}((\mathrm{Id}-\mathrm{A})U, V) = \mathrm{B}(F, V) \qquad \forall V =\in \mathbb{Y}_{0,\mathrm{N}}(\Gamma)\end{aligned} \tag{20}$$

Now we will consider only the case $U^{\mathrm{inc}} = [\,U_0^{\mathrm{inc}}(X) = \gamma^0(u_{\mathrm{inc}})\,;\, U_1^{\mathrm{inc}}(Y) = 0\,]$. Then

$$F = (\mathrm{Id}-\mathrm{A})U^{\mathrm{inc}} = \begin{bmatrix}\gamma^0(u_{\mathrm{inc}}) - \gamma^0 \cdot \mathrm{G}^0_\kappa(\gamma^0(u_{\mathrm{inc}}))\\ -\gamma^1 \cdot \mathrm{G}^0_\kappa(\gamma^0(u_{\mathrm{inc}}))\end{bmatrix} = \begin{bmatrix}\gamma^0(u_{\mathrm{inc}})\\ \gamma^1(u_{\mathrm{inc}})\end{bmatrix}$$

Let us clarify what $\mathrm{B}(U,V)$ is :

$$\mathrm{B}(U,V) = \mathrm{B}\Big(\begin{bmatrix}U_0\\ U_1\end{bmatrix}, \begin{bmatrix}V_0\\ V_1\end{bmatrix}\Big) = \mathrm{B}_0(U_0,V_0) \ + \ \mathrm{B}_1(U_1,V_1)$$

wich means $\mathrm{B}(U,V) = \int_{\partial\Omega_0} U_0^{\mathrm{D}}\, V_0^{\mathrm{N}}\, d\sigma - \int_{\partial\Omega_0} U_0^{\mathrm{N}}\, V_0^{\mathrm{D}}\, d\sigma + \int_{\partial\Omega_1} U_1^{\mathrm{D}}\, V_1^{\mathrm{N}}\, d\sigma - \int_{\partial\Omega_1} U_1^{\mathrm{N}}\, V_1^{\mathrm{D}}\, d\sigma$ and using (16) and (17) we obtain

$$\mathrm{B}(U,V) = 2\int_{\Gamma_{01}} U_0^{\mathrm{D}}\, V_0^{\mathrm{N}}\, d\sigma \ - \ \int_{\partial\Omega_0} U_0^{\mathrm{N}}\, V_0^{\mathrm{D}}\, d\sigma \ - \ \int_{\partial\Omega_1} U_1^{\mathrm{N}}\, V_1^{\mathrm{D}}\, d\sigma \tag{21}$$

Let us now clarify what $\mathrm{B}(\mathrm{A}U,V)$ is :

$$\mathrm{B}(\mathrm{A}U,V) = \mathrm{B}\Big(\begin{bmatrix}\gamma^0 \cdot G^0_\kappa\{U_0\} + \gamma^0 \cdot G^1_\kappa\{U_1\}\\ \gamma^1 \cdot G^0_\kappa\{U_0\} + \gamma^1 \cdot G^1_\kappa\{U_1\}\end{bmatrix}, \begin{bmatrix}V_0\\ V_1\end{bmatrix}\Big) = \sum_{i,j\,=\,0,1} \mathrm{B}_i(\gamma^i \cdot G^j_\kappa\{U_j\}, V_i)$$

wich means

$$\begin{aligned}\mathrm{B}(\mathrm{A}U,V) = \ &\underbrace{\sum_{i,j\,=\,0,1}\int_{\partial\Omega_i} \gamma^i_{\mathrm{D}} \cdot \mathrm{DL}^j_\kappa\{U_j^{\mathrm{D}}\}\, V_i^{\mathrm{N}}\, d\sigma}_{\boxed{1}} \ + \ \underbrace{\sum_{i,j\,=\,0,1}\int_{\partial\Omega_i} \gamma^i_{\mathrm{D}} \cdot \mathrm{SL}^j_\kappa\{U_j^{\mathrm{N}}\}\, V_i^{\mathrm{N}}\, d\sigma}_{\boxed{2}}\\ &\underbrace{-\sum_{i,j\,=\,0,1}\int_{\partial\Omega_i} \gamma^i_{\mathrm{N}} \cdot \mathrm{DL}^j_\kappa\{U_j^{\mathrm{D}}\}\, V_i^{\mathrm{D}}\, d\sigma}_{\boxed{3}} \ \underbrace{- \sum_{i,j\,=\,0,1}\int_{\partial\Omega_i} \gamma^i_{\mathrm{N}} \cdot \mathrm{SL}^j_\kappa\{U_j^{\mathrm{N}}\}\, V_i^{\mathrm{D}}\, d\sigma}_{\boxed{4}}\end{aligned}$$

with

$$\begin{aligned}\boxed{1} = \sum_{i,j\,=\,0,1}\int_{\partial\Omega_i} \gamma^i_{\mathrm{D}} \cdot \mathrm{DL}^j_\kappa\{U_j^{\mathrm{D}}\}\, V_i^{\mathrm{N}}\, d\sigma = \ &\int_{\partial\Omega_0} \gamma^0_{\mathrm{D}} \cdot \mathrm{DL}^0_\kappa\{U_0^{\mathrm{D}}\}\, V_0^{\mathrm{N}}\, d\sigma + \int_{\partial\Omega_0} \gamma^0_{\mathrm{D}} \cdot \mathrm{DL}^1_\kappa\{U_1^{\mathrm{D}}\}\, V_0^{\mathrm{N}}\, d\sigma\\ &+\int_{\partial\Omega_1} \gamma^1_{\mathrm{D}} \cdot \mathrm{DL}^0_\kappa\{U_0^{\mathrm{D}}\}\, V_1^{\mathrm{N}}\, d\sigma + \int_{\partial\Omega_1} \gamma^1_{\mathrm{D}} \cdot \mathrm{DL}^1_\kappa\{U_1^{\mathrm{D}}\}\, V_1^{\mathrm{N}}\, d\sigma\end{aligned}$$

and using (16) and (17)

$$\boxed{1} = 2\int_{\Gamma_{01}} \{\gamma_{\mathrm{D}}^{0}\} \cdot \mathrm{DL}_{\kappa}^{0}\{U_0^{\mathrm{D}}\}\, V_0^{\mathrm{N}}\, d\sigma \;+\; 2\int_{\Gamma_{01}} \{\gamma_{\mathrm{D}}^{1}\} \cdot \mathrm{DL}_{\kappa}^{1}\{U_1^{\mathrm{D}}\}\, V_1^{\mathrm{N}}\, d\sigma \tag{22}$$

We also obtain

$$\boxed{2} = 2\int_{\Gamma_{01}} \{\gamma_{\mathrm{D}}^{0}\} \cdot \mathrm{SL}_{\kappa}^{0}\{U_0^{\mathrm{N}}\}\, V_0^{\mathrm{N}}\, d\sigma \;+\; 2\int_{\Gamma_{01}} \{\gamma_{\mathrm{D}}^{1}\} \cdot \mathrm{SL}_{\kappa}^{1}\{U_1^{\mathrm{N}}\}\, V_1^{\mathrm{N}}\, d\sigma \tag{23}$$

$$\begin{aligned}\boxed{3} = & -\int_{\partial\Omega_0} \gamma_{\mathrm{N}}^{0} \cdot \mathrm{DL}_{\kappa}^{0}\{U_0^{\mathrm{D}}\}\, V_0^{\mathrm{D}}\, d\sigma \;-\; \int_{\partial\Omega_0} \gamma_{\mathrm{N}}^{0} \cdot \mathrm{DL}_{\kappa}^{1}\{U_1^{\mathrm{D}}\}\, V_0^{\mathrm{D}}\, d\sigma \\ & -\int_{\partial\Omega_1} \gamma_{\mathrm{N}}^{1} \cdot \mathrm{DL}_{\kappa}^{0}\{U_0^{\mathrm{D}}\}\, V_1^{\mathrm{D}}\, d\sigma \;-\; \int_{\partial\Omega_1} \gamma_{\mathrm{N}}^{1} \cdot \mathrm{DL}_{\kappa}^{1}\{U_1^{\mathrm{D}}\}\, V_1^{\mathrm{D}}\, d\sigma\end{aligned} \tag{24}$$

$$\begin{aligned}\boxed{4} = & -\int_{\partial\Omega_0} \gamma_{\mathrm{N}}^{0} \cdot \mathrm{SL}_{\kappa}^{0}\{U_0^{\mathrm{N}}\}\, V_0^{\mathrm{D}}\, d\sigma \;-\; \int_{\partial\Omega_0} \gamma_{\mathrm{N}}^{0} \cdot \mathrm{SL}_{\kappa}^{1}\{U_1^{\mathrm{N}}\}\, V_0^{\mathrm{D}}\, d\sigma \\ & -\int_{\partial\Omega_1} \gamma_{\mathrm{N}}^{1} \cdot \mathrm{SL}_{\kappa}^{0}\{U_0^{\mathrm{N}}\}\, V_1^{\mathrm{D}}\, d\sigma \;-\; \int_{\partial\Omega_1} \gamma_{\mathrm{N}}^{1} \cdot \mathrm{SL}_{\kappa}^{1}\{U_1^{\mathrm{N}}\}\, V_1^{\mathrm{D}}\, d\sigma\end{aligned} \tag{25}$$

4.2.2 Numerical experiments

In this paragraph we present numerical results obtained by testing a discretized formulation of (20) using a Petrov-Galerkin method we describe below.
Ω_2 is the unit disk and Ω_1 is an half annulus centered in the origin of small radius 1 and large radius 2.
One more time we use GMSH to generate a mesh of the domain's boundary, a C-library of integral equations (developed by Patrick Meury) to assembly operators and MATLAB to plot results and any usefull information. All the functions we add are written in C-language. We don't use any C-library of linear algebra.

Discretization For discretization, we considered a paneling $\Gamma^{\mathrm{h}} = \cup_{i=1}^{\mathrm{N}_\Gamma}\Gamma_i^{\mathrm{h}}$ of Γ where each Γ_i^{h} is a segment. Let us call $\mathrm{I}_\Gamma = \{1, 2, 3, \,..\, , \mathrm{N}_\Gamma\}$ the set of indexes of Γ^{h} nodes, size N_Γ.
We also consider a paneling $\Gamma_0^{\mathrm{h}} = \cup_{i=1}^{\mathrm{N}_0}\Gamma_{0,i}^{\mathrm{h}}$ of Γ_0 and $\mathrm{I}_0 \subset \mathrm{I}_\Gamma$ (size N_0), a paneling $\Gamma_1^{\mathrm{h}} = \cup_{i=1}^{\mathrm{N}_1}\Gamma_{1,i}^{\mathrm{h}}$ of Γ_1 and $\mathrm{I}_1 \subset \mathrm{I}_\Gamma$ (size N_1), a paneling $\Gamma_2^{\mathrm{h}} = \cup_{i=1}^{\mathrm{N}_2}\Gamma_{2,i}^{\mathrm{h}}$ of Γ_2 and $\mathrm{I}_2 \subset \mathrm{I}_\Gamma$ (size N_2) and a paneling $\Gamma_{01}^{\mathrm{h}} = \cup_{i=1}^{\mathrm{N}_{01}}\Gamma_{01,i}^{\mathrm{h}}$ of Γ_{01} and $\mathrm{I}_{01} = \mathrm{I}_0 \cap \mathrm{I}_1 \subset \mathrm{I}_\Gamma$ (size N_{01}).
Let us call $\mathrm{I}_{02} = \mathrm{I}_0 \cap \mathrm{I}_2$ and $\mathrm{I}_{12} = \mathrm{I}_1 \cap \mathrm{I}_2$. Of course we have $\mathrm{I}_\Gamma = \mathrm{I}_0 \cup \mathrm{I}_1 \cup \mathrm{I}_2$ and $\mathrm{I}_{01} \neq \varnothing$ (because $\Gamma_{01} \neq \varnothing$).

Then we considered the space $\mathbb{V}_h^{+\frac{1}{2}}(\Gamma_0)$ with

$$\mathbb{V}_h^{+\frac{1}{2}}(\Gamma_0) = \{v_h \in C^0(\Gamma_0) \mid v_h|_{\Gamma_{0,i}^{\mathrm{h}}} \in \mathbb{P}_1 \text{ for } \Gamma_{0,i}^{\mathrm{h}} \subset \Gamma_0^{\mathrm{h}},\ i = 1 \dots \mathrm{N}_0\} = \operatorname*{span}_{i\in \mathrm{I}_0}\{\Phi_0^i\}$$

$\mathbb{V}_h^{+\frac{1}{2}}(\Gamma_0)$ is an approximation space for $\mathrm{H}^{-\frac{1}{2}}(\Gamma_0)$ and $\mathrm{H}^{+\frac{1}{2}}(\Gamma_0)$. We also define $\mathbb{V}_h^{+\frac{1}{2}}(\Gamma_1)$ to approximate $\mathrm{H}^{-\frac{1}{2}}(\Gamma_1)$ and $\mathrm{H}^{+\frac{1}{2}}(\Gamma_1)$.

Let us call $\mathbb{X}^{\mathrm{h}}_{0,\mathrm{D}}(\Gamma)$ the approximtion space for $\mathbb{X}_{0,\mathrm{D}}(\Gamma)$.

$$U^{\mathrm{h}} = \begin{bmatrix} U_0^{\mathrm{h}}(X) \\ U_1^{\mathrm{h}}(Y) \end{bmatrix} = \begin{bmatrix} U_0^{\mathrm{h,D}}(X) \\ U_0^{\mathrm{h,N}}(X) \\ U_1^{\mathrm{h,D}}(Y) \\ U_1^{\mathrm{h,N}}(Y) \end{bmatrix} = \sum_{i=1}^{\mathrm{N}_\Gamma} \begin{bmatrix} \delta^i_{\mathrm{I}_0}\; U_0^{\mathrm{h,D}}(X_i)\; \Phi_0^i(X) \\ \delta^i_{\mathrm{I}_0}\; U_0^{\mathrm{h,N}}(X_i)\; \Phi_0^i(X) \\ \delta^i_{\mathrm{I}_1}\; U_1^{\mathrm{h,D}}(Y_i)\; \Phi_1^i(Y) \\ \delta^i_{\mathrm{I}_1}\; U_1^{\mathrm{h,N}}(Y_i)\; \Phi_1^i(Y) \end{bmatrix} \in \mathbb{X}^{\mathrm{h}}_{0,\mathrm{D}}(\Gamma)$$

$$\iff \begin{cases} \begin{bmatrix} U_0^{\mathrm{h,D}}(X) \\ U_0^{\mathrm{h,N}}(X) \\ U_1^{\mathrm{h,D}}(Y) \\ U_1^{\mathrm{h,N}}(Y) \end{bmatrix} \in \mathbb{V}_{\mathrm{h}}^{+\frac{1}{2}}(\Gamma_0) \times \mathbb{V}_{\mathrm{h}}^{+\frac{1}{2}}(\Gamma_0) \times \mathbb{V}_{\mathrm{h}}^{+\frac{1}{2}}(\Gamma_1) \times \mathbb{V}_{\mathrm{h}}^{+\frac{1}{2}}(\Gamma_1) \\ \\ U_0^{\mathrm{h,D}}(X_i) = U_1^{\mathrm{h,D}}(X_i) \qquad \forall i \in \mathrm{I}_{01} \\ U_0^{\mathrm{h,D}}(X_i) = 0 \qquad \forall i \in \mathrm{I}_{02} \\ U_1^{\mathrm{h,D}}(X_i) = 0 \qquad \forall i \in \mathrm{I}_{12} \\ U_0^{\mathrm{h,N}}(X_i) = -U_1^{\mathrm{h,N}}(X_i) \quad \forall i \in \mathrm{I}_{01} \end{cases}$$

Let us call $\mathbb{Y}^{\mathrm{h}}_{0,\mathrm{N}}(\Gamma)$ the approximtion space for $\mathbb{Y}_{0,\mathrm{N}}(\Gamma)$.

$$V^{\mathrm{h}} = \begin{bmatrix} V_0^{\mathrm{h}}(X) \\ V_1^{\mathrm{h}}(Y) \end{bmatrix} = \begin{bmatrix} V_0^{\mathrm{h,D}}(X) \\ V_0^{\mathrm{h,N}}(X) \\ V_1^{\mathrm{h,D}}(Y) \\ V_1^{\mathrm{h,N}}(Y) \end{bmatrix} = \sum_{i=1}^{\mathrm{N}_\Gamma} \begin{bmatrix} \delta^i_{\mathrm{I}_0}\; V_0^{\mathrm{h,D}}(X_i)\; \Phi_0^i(X) \\ \delta^i_{\mathrm{I}_0}\; V_0^{\mathrm{h,N}}(X_i)\; \Phi_0^i(X) \\ \delta^i_{\mathrm{I}_1}\; V_1^{\mathrm{h,D}}(Y_i)\; \Phi_1^i(Y) \\ \delta^i_{\mathrm{I}_1}\; V_1^{\mathrm{h,N}}(Y_i)\; \Phi_1^i(Y) \end{bmatrix} \in \mathbb{Y}^{\mathrm{h}}_{0,\mathrm{N}}(\Gamma)$$

$$\iff \begin{cases} \begin{bmatrix} V_0^{\mathrm{h,D}}(X) \\ V_0^{\mathrm{h,N}}(X) \\ V_1^{\mathrm{h,D}}(Y) \\ V_1^{\mathrm{h,N}}(Y) \end{bmatrix} \in \mathbb{V}_{\mathrm{h}}^{+\frac{1}{2}}(\Gamma_0) \times \mathbb{V}_{\mathrm{h}}^{+\frac{1}{2}}(\Gamma_0) \times \mathbb{V}_{\mathrm{h}}^{+\frac{1}{2}}(\Gamma_1) \times \mathbb{V}_{\mathrm{h}}^{+\frac{1}{2}}(\Gamma_1) \\ \\ V_0^{\mathrm{h,N}}(X_i) = V_1^{\mathrm{h,N}}(X_i) \qquad \forall i \in \mathrm{I}_{01} \\ V_0^{\mathrm{h,N}}(X_i) = 0 \qquad \forall i \in \mathrm{I}_{02} \\ V_1^{\mathrm{h,N}}(X_i) = 0 \qquad \forall i \in \mathrm{I}_{12} \\ V_0^{\mathrm{h,D}}(X_i) = -V_1^{\mathrm{h,D}}(X_i) \quad \forall i \in \mathrm{I}_{01} \end{cases}$$

Let us define the mass matrix. Using (21) and the definitions of $\mathbb{X}_{0,\mathrm{D}}(\Gamma)$ and $\mathbb{Y}_{0,\mathrm{N}}(\Gamma)$ we have

$$\begin{aligned} \mathrm{B}(U^{\mathrm{h}}, V^{\mathrm{h}}) \;&= 2\int_{\Gamma_{01}} U_0^{\mathrm{h,D}}(X)\, V_0^{\mathrm{h,N}}(X)\, d\sigma \;-\; \int_{\partial\Omega_0} U_0^{\mathrm{h,N}}(X)\, V_0^{\mathrm{h,D}}(X)\, d\sigma \;-\; \int_{\partial\Omega_1} U_1^{\mathrm{h,N}}(Y)\, V_1^{\mathrm{h,D}}(Y)\, d\sigma \\ &= 2 \sum_{i,j\,\in\,\mathrm{I}_{01}} U_0^{\mathrm{h,D}}(X_i)\, V_0^{\mathrm{h,N}}(X_j) \int_{\Gamma^{\mathrm{h}}_{01}} \Phi_0^i(X)\Phi_0^j(X)\, d\sigma \\ &- \sum_{i,j\,\in\,\mathrm{I}_0} U_0^{\mathrm{h,N}}(X_i)\, V_0^{\mathrm{h,D}}(X_j) \int_{\Gamma^{\mathrm{h}}_0} \Phi_0^i(X)\Phi_0^j(X)\, d\sigma - \sum_{i,j\,\in\,\mathrm{I}_1} U_1^{\mathrm{h,N}}(Y_i)\, V_1^{\mathrm{h,D}}(Y_j) \int_{\Gamma^{\mathrm{h}}_1} \Phi_1^i(Y)\Phi_1^j(Y)\, d\sigma \end{aligned}$$

Let us define two submatrix $\mathrm{M}^{\mathrm{D,N}}$ (size N_{01} x N_{01}) and $\mathrm{M}^{\mathrm{N,D}}$ (size N_{Γ} x N_{Γ}) with

$$\begin{aligned}
\mathrm{M}^{\mathrm{D,N}}_{i,j} &= 2\sum_{i,j\,\in\,\mathrm{I}_{01}} U_0^{\mathrm{h,D}}(X_i)\,V_0^{\mathrm{h,N}}(X_j)\int_{\Gamma^{\mathrm{h}}_{01}} \Phi_0^i(X)\Phi_0^j(X)\,d\sigma\\
\mathrm{M}^{\mathrm{N,D}}_{i,j} &= -\sum_{i,j\,\in\,\mathrm{I}_{0}} U_0^{\mathrm{h,N}}(X_i)\,V_0^{\mathrm{h,D}}(X_j)\int_{\Gamma^{\mathrm{h}}_{0}} \Phi_0^i(X)\Phi_0^j(X)\,d\sigma\\
&\quad -\sum_{i,j\,\in\,\mathrm{I}_{1}} U_1^{\mathrm{h,N}}(Y_i)\,V_1^{\mathrm{h,D}}(Y_j)\int_{\Gamma^{\mathrm{h}}_{1}} \Phi_1^i(Y)\Phi_1^j(Y)\,d\sigma
\end{aligned}$$

Then the mass matrix M (size $(\mathrm{N}_{01}+\mathrm{N}_{\Gamma})$ x $(\mathrm{N}_{01}+\mathrm{N}_{\Gamma})$) is defined by

$$\mathrm{M} = \begin{bmatrix}\mathrm{M}^{\mathrm{D,N}} & 0\\ 0 & \mathrm{M}^{\mathrm{N,D}}\end{bmatrix}$$

Then to derivate the discretized formulation of (20), we define 4 discrete operators :
A1 (size N_{01} x N_{01}), A2 (size N_{01} x N_{Γ}), A3 (size N_{Γ} x N_{01}) and A4 (size N_{Γ} x N_{Γ}) using respectively the continuous expressions (22), (23), (24) and (25). Setting

$$\mathrm{A} = \begin{bmatrix}\mathrm{A}_1 & \mathrm{A}_2\\ \mathrm{A}_3 & \mathrm{A}_4\end{bmatrix}\ ,\ \mathbb{U} = \begin{bmatrix}U_0^{\mathrm{h,D}}(X_i) = U_1^{\mathrm{h,D}}(X_i) & i\in\mathrm{I}_{01}\\ U^{\mathrm{h,N}}(X_j) & j\in\mathrm{I}_{\Gamma}\end{bmatrix}\ \text{ and }\ \mathbb{F} = \begin{bmatrix}\gamma_0^{\mathrm{D}}(u_{\mathrm{inc}})(X_i) = \gamma_1^{\mathrm{D}}(u_{\mathrm{inc}})(X_i) & i\in\mathrm{I}_{01}\\ \gamma^{\mathrm{N}}(u_{\mathrm{inc}})(X_j) & j\in\mathrm{I}_{\Gamma}\end{bmatrix}$$

the discretized formulation of (20) is

$$\begin{aligned}
&\text{Find } \ U^{\mathrm{h}}\in\mathbb{X}_{0,\mathrm{D}}(\Gamma)\quad\text{such that}\\
&(\mathrm{M}-\mathrm{A})\,\mathbb{U} = \mathrm{M}\,\mathbb{F}
\end{aligned}\tag{26}$$

To be very precise, the mass matrices in the left and the right side are not exactly the same : $\mathrm{M}^{\mathrm{N,D}}$ parts are the same but the $\mathrm{M}^{\mathrm{D,N}}$ parts are different around the triple points.
Finally, separating real and imaginary parts, the discretized formulation of (20) is

$$\begin{aligned}
&\text{Find } \ U^{\mathrm{h}}\in\mathbb{X}_{0,\mathrm{D}}(\Gamma)\quad\text{such that}\\
&\begin{pmatrix}\mathrm{M}-\Re e(\mathrm{A}) & \Im m(\mathrm{A})\\ -\Im m(\mathrm{A}) & \mathrm{M}-\Re e(\mathrm{A})\end{pmatrix}\begin{pmatrix}\Re e(\mathbb{U})\\ \Im m(\mathbb{U})\end{pmatrix} = \begin{pmatrix}\mathrm{M}\,\Re e(\mathbb{F})\\ \mathrm{M}\,\Im m(\mathbb{F})\end{pmatrix}
\end{aligned}\tag{27}$$

Mesh generation We use GMSH to create the mesh. In **Fig.6**, **Fig.7** and **Fig.8** we represent the mesh with GMSH and MATLAB.

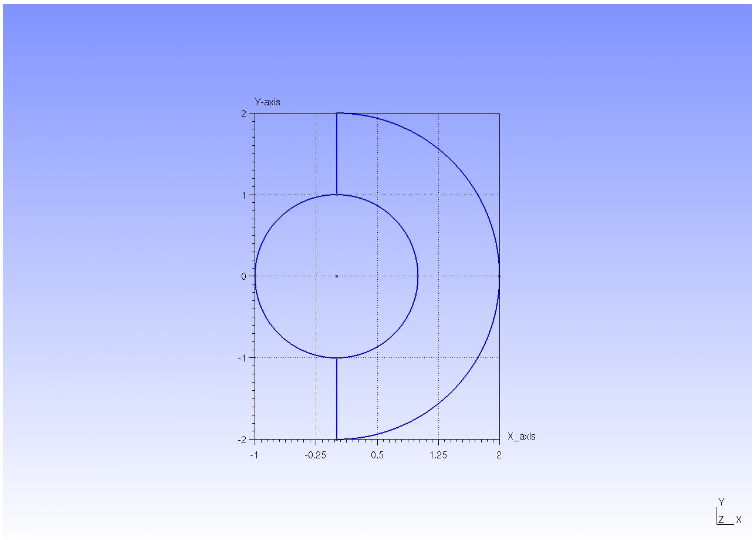

Fig.6 Γ displayed on GMSH

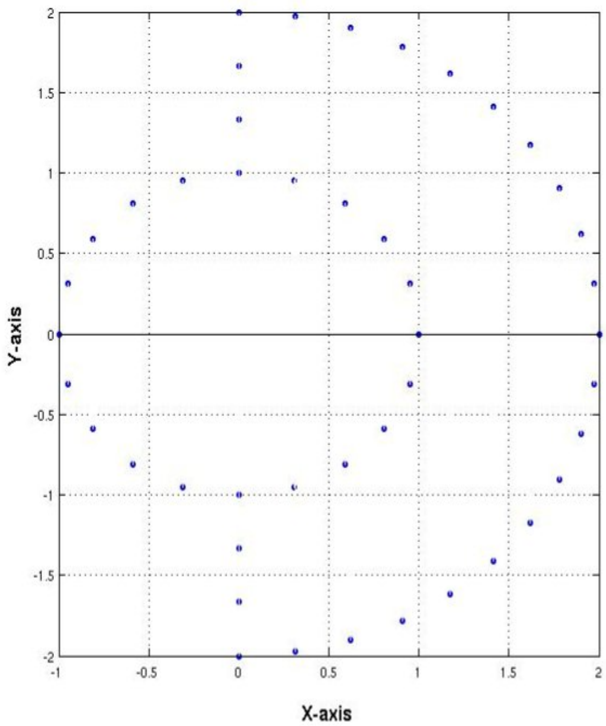

Fig.7 Mesh of Γ ploted on MATLAB, step h = 0.2

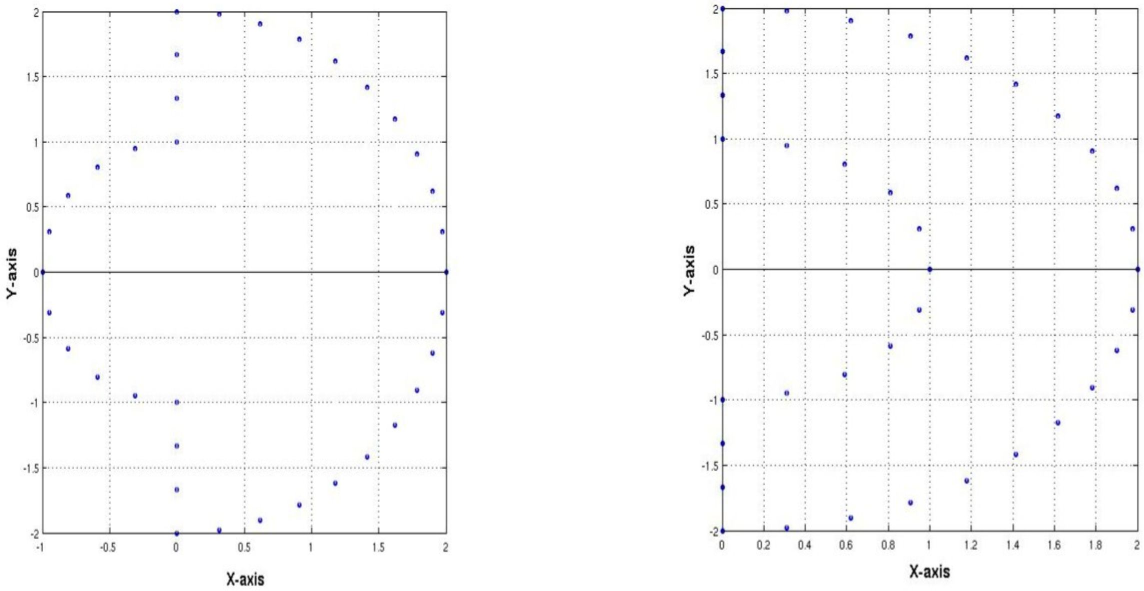

Fig.8 Mesh of $\partial\Omega_0$ and $\partial\Omega_1$ ploted on MATLAB, step h = 0.2

Test on Γ_2 The goal is to assemble just a part of the formulation (26) in order to refind the formulation (10). So we will ony use A4 and $\mathrm{M}^{\mathrm{N,D}}$: more precisely the restrictions $\mathrm{A4}|_{\Gamma_2}$ (size N_2 x N_2) of A4 and $\mathrm{M}^{\mathrm{N,D}}|_{\Gamma_2}$ (size N_2 x N_2) of $\mathrm{M}^{\mathrm{N,D}}$. The equivalent formulation of (10) using (26), with the notations we used for (10), is

$$\begin{aligned} &\text{Find } \; q \in \mathbb{V}_h^{+\frac{1}{2}}(\Gamma_2) \quad \text{such that} \\ &(-\frac{\mathrm{M}^{\mathrm{N,D}}|_{\Gamma_2}}{2} + \mathrm{A4}|_{\Gamma_2})\, Q = -\mathrm{M}^{\mathrm{N,D}}|_{\Gamma_2} \;\; Q_{\mathrm{inc}} \end{aligned} \tag{28}$$

In **Fig.9** we recall what Γ_2 is.

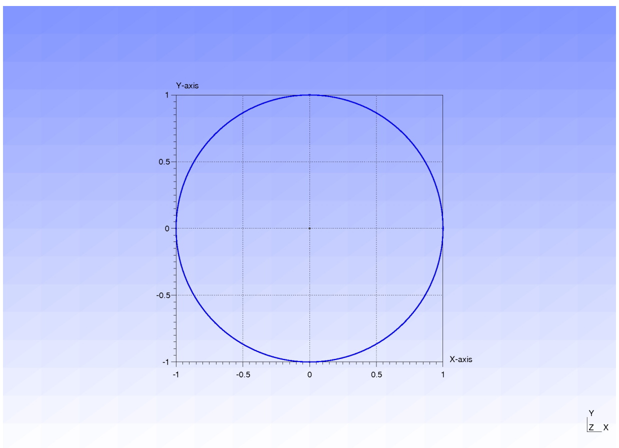

Fig.9 Γ_2 displayed on GMSH

We compute the same Q_{inc} as the one we computed for (10). We show below the results we have obtained for (28). q is the exact solution of our problem (components q_i) ; let us call q^{h} the approximate solution (components q_i^{h}). In **Fig.10** and **Fig.11** we represent the quadratic errors about real part and imaginary part as functions of the step of the mesh, for two values of κ (in reality we represent quadratic errors multiplied by $\text{h}^{-\frac{1}{2}}$ to simulate $\text{L}^2(\Gamma)$ errors).

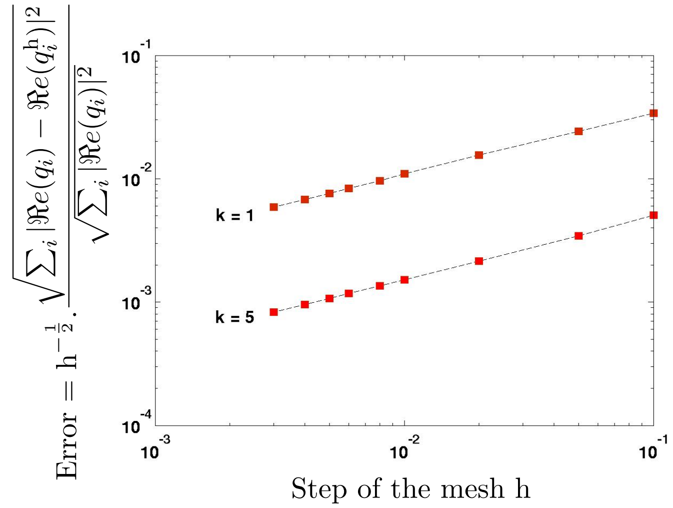

Fig.10 Error about real part for two values of κ

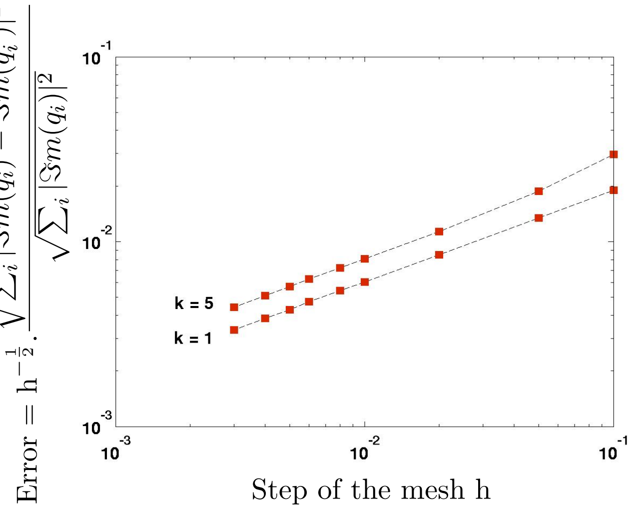

Fig.11 Error about imaginary part for two values of κ

In figure **Fig.12** we represent the condition number of the matrix associated to (28) as a function of the step of the mesh, for $\kappa = 2$.

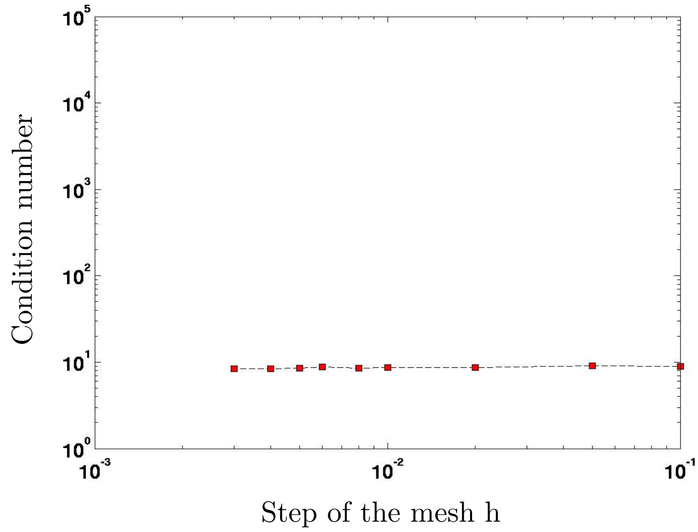

Fig.12 Stable conditioning

The condition number is one more time bounded independently of the step of the mesh.

Test on Γ_1 Let us consider $u \in \mathrm{H}^1(\Delta, \Omega_1)$ defined by

$$u(r) = \frac{\mathrm{H}_0^{(1)}(\kappa r)}{\mathrm{H}_0^{(1)}(\kappa)} - \frac{\mathrm{J}_0(\kappa r)}{\mathrm{J}_0(\kappa)}$$

Let us call $U_1 = [U_1^{\mathrm{D}}, U_1^{\mathrm{N}}] = \gamma^1(u)$. Because u is a solution of the Helmotz equation in Ω_1 we have $\gamma^1 \cdot \mathrm{G}_\kappa^1(U_1) = U_1$. The goal here is to assemble the operator $\gamma^1 \cdot \mathrm{G}_\kappa^1$ using parts of the operators computed for (26) and then test

$$\gamma^1 \cdot \mathrm{G}_\kappa^1(U_1) = U_1 \tag{29}$$

To do that, we only have to keep the parts with 1 (and only 1) in (22), (23), (24) and (25) : it defines new operators A1, A2, A3 and A4. Then, we take the restrictions $\mathrm{A2}|_{\Gamma_{01} \text{ x } \Gamma_1}$ (size N_{01} x N_1) of A2, $\mathrm{A3}|_{\Gamma_1 \text{ x } \Gamma_{01}}$ (size N_1 x N_{01}) of A3 and $\mathrm{A4}|_{\Gamma_1}$ (size N_1 x N_1) of A4.

Concerning the mass, we have to take one half of $\mathrm{M}^{\mathrm{D,N}}$ and only the part on Γ_1 for $\mathrm{M}^{\mathrm{N,D}}$ and then the restriction $\mathrm{M}^{\mathrm{N,D}}|_{\Gamma_1}$ (size N_1 x N_1).

Then the operator $\gamma^1 \cdot \mathrm{G}_\kappa^1$ can be discretized by

$$\begin{bmatrix} \frac{\mathrm{M}^{\mathrm{D,N}}}{2} + \mathrm{A1} & \mathrm{A2}|_{\Gamma_{01} \text{ x } \Gamma_1} \\ & \\ -\mathrm{A3}|_{\Gamma_1 \text{ x } \Gamma_{01}} & -\frac{\mathrm{M}^{\mathrm{N,D}}}{2} - \mathrm{A4}|_{\Gamma_1} \end{bmatrix}$$

Let us make explicit U_1^{D} and U_1^{N}

$$U_1^{\mathrm{D}}(r) = \frac{\mathrm{J}_0(\kappa r)\mathrm{J}_0(\kappa) + \mathrm{Y}_0(\kappa r)\mathrm{Y}_0(\kappa)}{|\mathrm{H}_0^{(1)}(\kappa)|^2} - \frac{\mathrm{J}_0(\kappa r)}{\mathrm{J}_0(\kappa)} + i\frac{\mathrm{Y}_0(\kappa r)\mathrm{J}_0(\kappa) - \mathrm{J}_0(\kappa r)\mathrm{Y}_0(\kappa)}{|\mathrm{H}_0^{(1)}(\kappa)|^2}$$

$$U_1^{\mathrm{N}}(r) = \begin{cases} \kappa\frac{\mathrm{J}_1(\kappa)\mathrm{J}_0(\kappa) + \mathrm{Y}_1(\kappa)\mathrm{Y}_0(\kappa)}{|\mathrm{H}_0^{(1)}(\kappa)|^2} - \kappa\frac{\mathrm{J}_1(\kappa)}{\mathrm{J}_0(\kappa)} - i\frac{2}{\pi|\mathrm{H}_0^{(1)}(\kappa)|^2} \ \text{ if } \ r = 1, \ \text{on } \Gamma_{21} \\ \\ -\kappa\frac{\mathrm{J}_1(2\kappa)\mathrm{J}_0(\kappa) + \mathrm{Y}_1(2\kappa)\mathrm{Y}_0(\kappa)}{|\mathrm{H}_0^{(1)}(\kappa)|^2} + \kappa\frac{\mathrm{J}_1(2\kappa)}{\mathrm{J}_0(\kappa)} + i\kappa\frac{-\mathrm{Y}_1(2\kappa)\mathrm{J}_0(\kappa) + \mathrm{J}_1(2\kappa)\mathrm{Y}_0(\kappa)}{|\mathrm{H}_0^{(1)}(\kappa)|^2} \ \text{ if } r = 2, \ \text{on } \Gamma_{01} \\ \\ 0 \ \text{ if } \ r = 1, \ \text{on } \Gamma_{01} \end{cases}$$

We show below the results we have obtained for (29). F_1 (:=MU_1) is the exact solution of our problem (components F_1^i) ; let us call F_1^{h} the approximate solution (components $F_1^{i,\mathrm{h}}$). In **Fig.13** and **Fig.14** we represent the quadratic errors about real part and imaginary part of the Dirichlet part as functions of the step of the mesh, for two values of κ (in reality we represent quadratic errors multiplied by $\mathrm{h}^{\frac{1}{2}}$ to simulate $\mathrm{L}^2(\Gamma)$ errors).

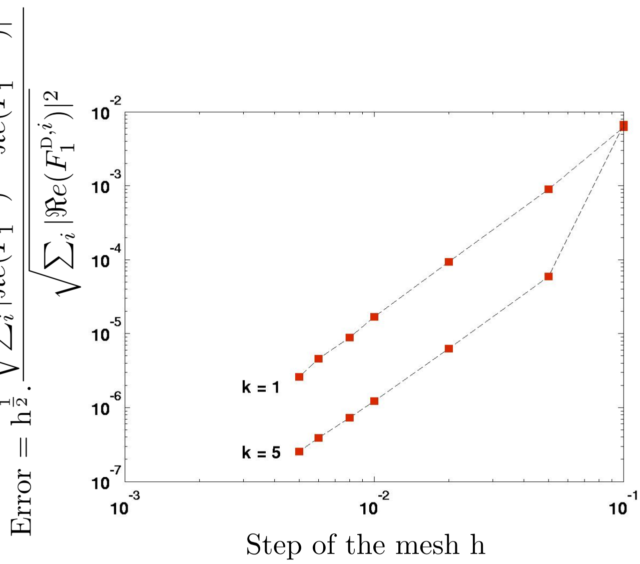

Fig.13 Error about real part of the Dirichlet part for two values of κ

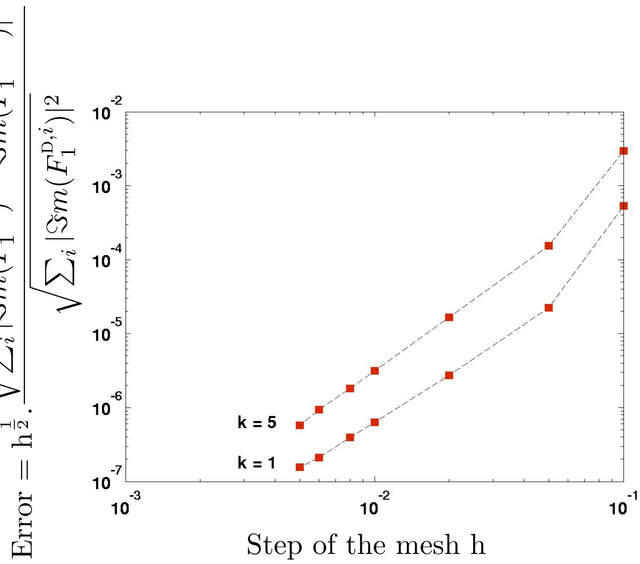

Fig.14 Error about imaginary part of the Dirichlet part for two values of κ

In **Fig.15** and **Fig.16** we represent the quadratic errors about real part and imaginary part of the Neumann part as functions of the step of the mesh, for two values of κ (in reality we represent quadratic errors multiplied by $\text{h}^{-\frac{1}{2}}$ to simulate $\text{L}^2(\Gamma)$ errors).

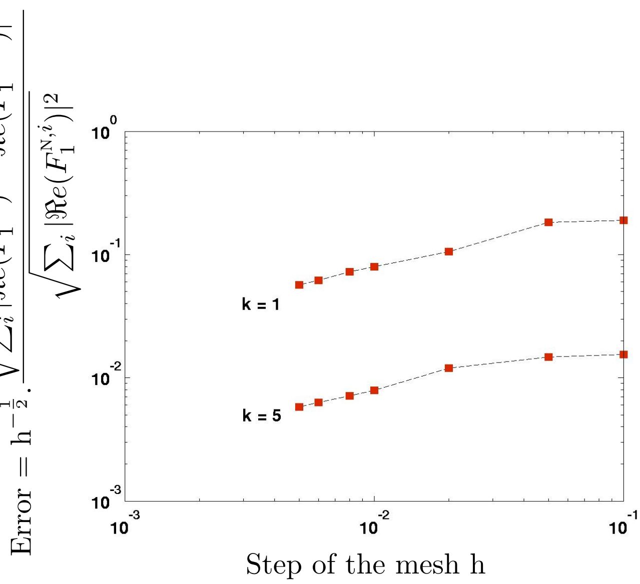

Fig.15 Error about real part of the Neumann part for two values of κ

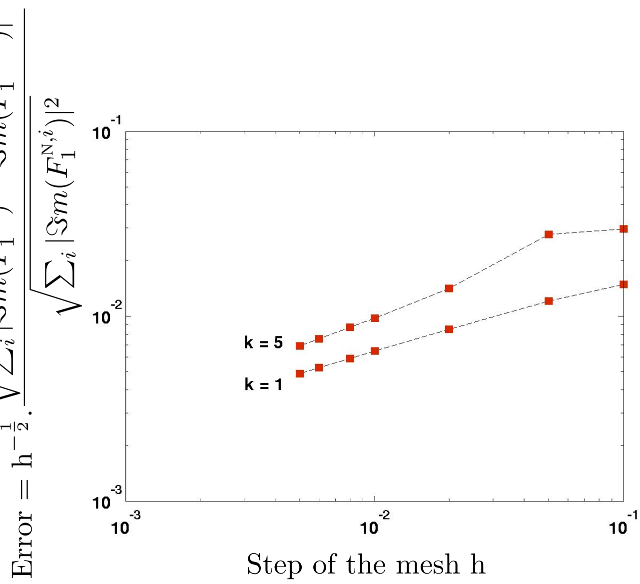

Fig.16 Error about imaginary part of the Neumann part for two values of κ

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References

[1] A. Bendali, M'B. Fares, and J. Gay. A boundary-element solution of the Leontovitch problem. IEEE Trans. Antennas and Propagation, 47(10):1597–1605, 1999.

[2] A. Buffa. Remarks on the discretization of some noncoercive operator with applications to the heterogeneous Maxwell equations. SIAM J. Numer. Anal., 43(1):1–18, 2005.

[3] X. Claeys. A single trace integral formulation of the second kind for acoustic scattering. Technical Report no. 2011-14, Seminar of Applied Mathematics, ETH, 2011.

[4] D. Colton and R. Kress. Inverse acoustic and electromagnetic scattering theory, volume 93 of Applied Mathematical Sciences. Springer-Verlag, Berlin, second edition, 1998.

[5] R. Hiptmair and C. Jerez-Hanckes. Multiple traces boundary integral formulation for helmholtz transmission problems. Technical Report no. 2010-35, Seminar of Applied Mathematics, ETH, 2010.

[6] W. McLean. Strongly elliptic systems and boundary integral equations. Cambridge University Press, Cambridge, 2000.

[7] T. von Petersdorff. Boundary Integral Equations for Mixed Dirichlet, Neumann and Transmission Problems. Math. Met. App. Sc., 11:185–213, 1989.